

*To the memory of Julius Wess*

# Gauge theory in deformed $\mathcal{N}=(1,1)$ superspace

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## Abstract

We review the non-anticommutative  $Q$ -deformations of  $\mathcal{N}=(1,1)$  supersymmetric theories in four-dimensional Euclidean harmonic superspace. These deformations preserve chirality and harmonic Grassmann analyticity. The associated field theories arise as a low-energy limit of string theory in specific backgrounds and generalize the Moyal-deformed supersymmetric field theories. A characteristic feature of the  $Q$ -deformed theories is the half-breaking of supersymmetry in the chiral sector of the Euclidean superspace. Our main focus is on the chiral singlet  $Q$ -deformation, which is distinguished by preserving the  $SO(4) \sim Spin(4)$  “Lorentz” symmetry and the  $SU(2)$   $R$ -symmetry. We present the superfield and component structures of the deformed  $\mathcal{N}=(1,0)$  supersymmetric gauge theory as well as of hypermultiplets coupled to a gauge superfield: invariant actions, deformed transformation rules, and so on. We discuss quantum aspects of these models and prove their renormalizability in the abelian case. For the charged hypermultiplet in an abelian gauge superfield background we construct the deformed holomorphic effective action.

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<sup>1</sup>Permanent address

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# 1 Introduction

By now, the concept of supersymmetry has been organically incorporated into modern high-energy theoretical physics. Originally, it was introduced at the mathematical level as a possible kind of new symmetry which extends the standard space-time symmetries by spinorial generators and relates bosons and fermions. Since then, the consequences of the supersymmetry hypothesis for particle physics has proved so fruitful that today it is hardly possible to doubt its validity. At present, the quest for supersymmetric partners of the known elementary particles is one of the main occupations of the forthcoming LHC experiments <sup>2</sup>.

Let us mention the most impressive achievements of supersymmetry. First of all, it yields a unified setup for describing bosons and fermions. In the Standard Model, it suggests a natural solution of the hierarchy problem. In grand unification models, it predicts the single-point meeting of the three basic running couplings (see e.g. [2]) and solves the problem of the proton lifetime. Finally, the most popular candidate for unifying gravity with quantum physics, String Theory, is to large extent based on the concept of supersymmetry. Supersymmetric theories in various dimensions originate from the low-energy limit of string theory with an appropriate choice of background manifold. New applications of supersymmetry regularly appear in various areas. The present review is devoted to a recent such development.

We will be concerned only with four-dimensional supersymmetric theories. The algebra of Poincaré supersymmetry in 4D Minkowski space is characterized by the number  $\mathcal{N}$  of fermionic spinorial generators.  $\mathcal{N}=1$  supersymmetry is referred to as *simple*, featuring only two two-component spinorial generators  $Q_\alpha$  and  $\bar{Q}_{\dot{\alpha}}$ . The spinorial generators of *extended* supersymmetry (with  $\mathcal{N}>1$ ) carry an index  $k$  of the fundamental representation of the  $R$ -symmetry group  $SU(\mathcal{N})$ . To date, the  $\mathcal{N}=1$  supersymmetric theories have been studied most thoroughly, both at the classical and at the quantum level, due to the existence of well established superfield techniques (see e.g. [3, 4]). Furthermore, only  $\mathcal{N}=1$  theories are really interesting for phenomenological applications. On the other hand, theories with extended supersymmetry exhibit quite remarkable and unique properties. For instance,  $\mathcal{N}=2$  supersymmetry imposes so severe constraints on the quantum dynamics that it becomes possible to find exact expressions (and values) for some important quantities. It is known that  $\mathcal{N}=2$  supersymmetric theories are one-loop exact due to so-called “non-renormalization” theorems. Moreover, the low-energy quantum effective action in  $\mathcal{N}=2$  supersymmetric gauge theory can be exactly evaluated non-perturbatively (so-called Seiberg-Witten theory [5]). However, among the supersymmetric field theories, the unique place belongs to the  $\mathcal{N}=4$  supergauge model. It possesses the maximal number of supersymmetries admitting spins not higher than one. The restrictions of  $\mathcal{N}=4$  supersymmetry on the quantum structure of this theory turn out to be so strong that they ensure ultraviolet finiteness of this theory (i.e. it contains no quantum divergences at all). Also  $\mathcal{N}=4$  supergauge theory is most intimately related to superstring theory, e.g. via the renowned AdS/CFT correspondence (see reviews [6]).

Since there is no experimental evidence for supersymmetry at the energies achievable by now, we must assume it to be broken, leading to the problem of appropriate theoretical mechanisms for such breaking. One possibility is the so-called soft breaking of supersymmetry. It is used in supersymmetric gauge theories and adds to the action certain mass terms which preserve gauge invariance but break supersymmetry. If supersymmetry is spontaneously broken, auxiliary

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<sup>2</sup>The phenomenological aspects of supersymmetry are discussed in detail e.g. in [1].

fields develop non-vanishing vacuum values and spinorial Goldstone fields (goldstini) appear. Standard methods of supersymmetry breaking can ruin the remarkable quantum properties of supersymmetric theories or, at least, limit the range of their applicability. Therefore, the search for and study of alternative supersymmetry breaking schemes are of clear importance.

A new mechanism for breaking space-time symmetries in quantum field theory arises from the hypothesis of noncommutativity of the space-time coordinates,

$$[x^m, x^n] = i\vartheta^{mn} = \text{const.} \quad (1.1)$$

Here, the constants  $\vartheta^{mn}$  are the parameters of the deformation of the commutative algebra of functions given on standard Minkowski space with coordinates  $x^m$ . In the noncommutative field theory based on the relation (1.1) [8, 9], Lorentz invariance is broken but translation invariance is still alive. On general findings, noncommutativity is implemented by inserting the so-called  $\star$ -product everywhere. In the case of deformation (1.1) the  $\star$ -multiplication on fields is realized with the help of the pseudo-differential operator  $P$  (the Poisson structure operator):

$$\phi(x) \star \psi(x) = \phi e^P \psi \quad \text{where} \quad P = \frac{i}{2} \overleftarrow{\partial}_m \vartheta^{mn} \overrightarrow{\partial}_n. \quad (1.2)$$

For constructing the classical action of noncommutative theories it suffices to replace the standard multiplication of the fields in the undeformed Lagrangian by the  $\star$ -multiplication (1.2). In this approach, the free part of the action preserves Lorentz invariance, while the breaking of Lorentz invariance due to the deformation comes out only in the interactions.

The relation (1.1) can be employed also to deform a superspace. However, the noncommutativity of bosonic coordinates alone does not trigger any breaking of supersymmetry. Formally, one can deform the algebra of both even and odd coordinates in superspace (see e.g. [7]). For Minkowski signature, however, the deformation of the fermionic superspace coordinates is not very well elaborated since it is very difficult to simultaneously maintain reality,  $\star$ -product associativity, and the preservation of chiral supersymmetry representations in the noncommutative theory (some attempts to overcome this problem were recently undertaken in [10] and [11]).

In the Euclidean version of  $\mathcal{N}=1$  superspace, in contrast, the Grassmann-odd coordinates  $\theta^\alpha$  and  $\bar{\theta}^{\dot{\alpha}}$  are not related by complex conjugation. We speak of Euclidean  $\mathcal{N}=(n/2, n/2)$  supersymmetry denoting the number of left-chiral and right-chiral (antichiral) spinorial generators in the superalgebra. In  $\mathcal{N}=(1/2, 1/2)$  supersymmetric theories formulated in Euclidean superspace it is therefore consistent to deform the left-chiral fermionic coordinates [12],

$$\{\theta^\alpha, \theta^\beta\} = C^{\alpha\beta} = \text{const} \quad \text{while} \quad \{\bar{\theta}^{\dot{\alpha}}, \bar{\theta}^{\dot{\beta}}\} = \{\theta^\alpha, \bar{\theta}^{\dot{\beta}}\} = 0, \quad (1.3)$$

thereby replacing a Grassmann algebra with a Clifford algebra. The parameters  $C^{\alpha\beta}$  deform the algebra of functions on  $\mathcal{N}=(1/2, 1/2)$  superspace. The remaining (anti)commutativity relations in the chiral basis are not altered, in order to preserve chirality. If such a non-anticommutative deformation is introduced exclusively in the left-chiral sector of the superspace, the original  $\mathcal{N}=(1/2, 1/2)$  Euclidean supersymmetry gets broken to  $\mathcal{N}=(1/2, 0)$ . It is obvious that the opportunity of such a half-breaking of supersymmetry exists only in Euclidean superspace.

We point out that the existence of non-anticommutative deformations preserving chirality derives from superstring theory [12, 13, 14]. Since the spectrum of IIB supergravity contains the four-form potential, the  $\mathcal{N}=(1, 1)$  superstring provides a self-dual five-form field-strength

background, which in first approximation is assumed to be constant. After a compactification to the orbifold  $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$  one obtains a four-dimensional  $\mathcal{N}=(1, 1)$  superstring in the background of a constant selfdual graviphoton field strength  $F^{\alpha\beta}$  (with  $F^{\dot{\alpha}\dot{\beta}} = 0$ , i.e. one considers Euclidean space). It turns out that the correlation functions  $\langle \theta^\alpha(\tau) \theta^\beta(\tau') \rangle$  are proportional to the constant field  $F^{\alpha\beta}$ , whereas those involving the conjugate variables  $\bar{\theta}^{\dot{\alpha}}$  are trivial. In the effective low-energy field theory, such string variables become fermionic coordinates of a superspace with precisely the non-trivial anticommutation relations (1.3). String models in the background of a constant self-dual gauge field can have interesting phenomenological properties. For instance, it was shown in [14] that the gluon potential in  $\mathcal{N}=(1/2, 1/2)$  supersymmetric theories can be modified by a non-anticommutative deformation such as to acquire a nontrivial vacuum expectation value, which may be related to quark confinement.

Field theories defined in Euclidean superspaces with deformed anticommutation relations of the type (1.3) are referred to as  $\mathcal{N}=1/2$  (or  $\mathcal{N}=(1/2, 0)$ ) non-anticommutative theories. These theories possess a number of attractive properties. For example, it was established in [15, 16, 17, 18, 19] that the  $\mathcal{N}=1/2$  supersymmetric Wess-Zumino model and the  $\mathcal{N}=1/2$  supersymmetric gauge theory inherit the renormalizability of their undeformed prototypes. In the Lagrangians of these models, non-anticommutative deformations (1.3) give rise to additional terms polynomial in the deformation parameters  $C^{\alpha\beta}$ . These terms can be treated as new interaction vertices, the powers of  $C^{\alpha\beta}$  playing the role of coupling constants with negative mass dimension. According to the standard lore of quantum field theory, vertices should give rise to non-renormalizable divergences. However, the extra terms brought into the action by the non-anticommutative deformations appear in a non-symmetric way (they are not accompanied by similar terms with  $C^{\dot{\alpha}\dot{\beta}}$ ), and the renormalization of such theories requires special analysis. For instance, in [15, 16] it was found that, a single new term was generated at quantum level, and the non-anticommutative Wess-Zumino model is multiplicatively renormalizable. For the  $\mathcal{N}=1/2$  super Yang-Mills model it was shown [19] that all new divergences owed to the non-anticommutative deformation can be eliminated by a shift of one spinor field. As a result of these studies, all considered  $\mathcal{N}=1/2$  theories were found to be renormalizable and, hence, may be of phenomenological interest (after performing a Wick rotation to the Minkowski signature). Furthermore, the effective action of the  $\mathcal{N}=1/2$  supersymmetric Wess-Zumino model and the Yang-Mills theory was studied in [20].

Let us turn to the extended supersymmetry and its non-anticommutative deformation. We consider Euclidean  $\mathcal{N}=(1, 1)$  superspace with Grassmann coordinates  $\theta_i^\alpha, \bar{\theta}^{\dot{\alpha}j}$  with  $i, j = 1, 2$  and  $\alpha = 1, 2$  and  $\dot{\alpha} = \dot{1}, \dot{2}$ . The left-chiral deformation (1.3) generalizes to [21, 22]

$$\{\hat{\theta}_i^\alpha, \hat{\theta}_j^\beta\} = C_{ij}^{\alpha\beta} = \text{const}, \quad (1.4)$$

with all other (anti)commutation relations between chiral coordinates of the  $\mathcal{N}=(1, 1)$  superspace remaining undeformed. The constant tensor  $C_{ij}^{\alpha\beta}$  decomposes into irreducible pieces,

$$C_{ij}^{\alpha\beta} = C_{(ij)}^{(\alpha\beta)} + \varepsilon^{\alpha\beta} \varepsilon_{ij} I. \quad (1.5)$$

Putting all components but  $C_{11}^{\alpha\beta}$  to zero, we recover the deformation (1.3). Various types of such deformations were studied in [23, 24, 25]. Of particular interest is the pure-trace deformation  $C_{ij}^{\alpha\beta} = \varepsilon^{\alpha\beta} \varepsilon_{ij} I$ . This type was named non-anticommutative chiral singlet deformation [21, 22]. Since the chiral singlet deformation is fully specified by a single parameter  $I$  which carries

no indices, it does not break Euclidean  $SO(4)$  invariance or  $SU(2)$   $R$ -symmetry. However, it breaks  $\mathcal{N}=(1,1)$  supersymmetry down to  $\mathcal{N}=(1,0)$ . Apart from these special properties, chiral singlet deformations can be given a stringy interpretation [26]. Unlike the  $\mathcal{N}=(1/2,0)$  case, for deriving chiral singlet deformations one must consider the  $\mathcal{N}=4$  superstring in the background of a constant axion field strength compactified on the orbifold  $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$ . The stringy origin of non-singlet deformations of  $\mathcal{N}=(1,1)$  supersymmetry was discussed in [27].

Like for the bosonic deformation (1.1), the relations (1.3) are also implemented in terms of an appropriate  $\star$ -product, which now operates on superfunctions of the coordinates of the undeformed  $\mathcal{N}=(1/2, 1/2)$  superspace  $z = (x^m, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$ :

$$A(z) \star B(z) = A e^{P_C} B \quad \text{with} \quad P_C = -\frac{1}{2} \overleftarrow{Q}_\alpha C^{\alpha\beta} \overrightarrow{Q}_\beta, \quad (1.6)$$

where  $Q_\alpha$  are the left-chiral supercharges. In the chiral basis the generators  $Q_\alpha$  coincide with the partial derivatives with respect to the left Grassmann coordinates,  $Q_\alpha = \partial_\alpha$ . Non-anticommutative models with simple supersymmetry are obtained from the corresponding undeformed models via insertion of the  $\star$ -multiplication (1.6) everywhere inside the corresponding superfield Lagrangians [12]. The criterion of preserving some symmetry of the “classical” (undeformed) action in the non-anticommutative case is the commuting of the symmetry generator with the Poisson operator  $P_C$  in (1.6).

The expression (1.6) for the  $\star$ -product can easily be generalized to the extended supersymmetry (1.4) [21, 22]:

$$A(z) \star B(z) = A e^{P_C} B \quad \text{with} \quad P_C = -\overleftarrow{Q}_\alpha^i C_{ij}^{\alpha\beta} \overrightarrow{Q}_\beta^j. \quad (1.7)$$

Here, the  $\mathcal{N}=(1,0)$  supersymmetry generators  $Q_\alpha^i$  can be chosen in the chiral basis,  $Q_\alpha^i = \partial/\partial\theta_\alpha^i \equiv \partial_\alpha^i$ , and  $A(z)$  and  $B(z)$  are arbitrary superfunctions on the extended superspace  $z = (x^m, \theta_\alpha^i, \bar{\theta}^{\dot{\alpha}j})$ . The most appropriate superfield formulation of models with  $\mathcal{N}=(1,1)$  supersymmetry is provided by the harmonic superspace approach, which has been worked out in detail for  $\mathcal{N}=2$  supersymmetric theories in Minkowski space [34, 35]. This approach allows one to write down superfield actions for non-anticommutative models in manifestly  $\mathcal{N}=(1,0)$  supersymmetric form, and it also ensures the preservation of supersymmetry at all stages of the quantum calculations. Non-anticommutative deformations of the type (1.4) for harmonic superspace were introduced in [21, 22], while non-anticommutative  $\mathcal{N}=(1,0)$  models of hypermultiplets and gauge superfields were introduced and studied in [21, 22, 25, 26, 29, 30, 31]. In these papers, the component structure of the corresponding classical deformed actions has been established.

The Poisson operators  $P_C$  generating the non-anticommutative deformations (1.6) and (1.7) are composed from the supercharges of the unbroken  $\mathcal{N}=(1/2, 0)$  or  $\mathcal{N}=(1, 0)$  supersymmetries, respectively, hence such deformations are called  $Q$ -deformations. By definition, the operators  $P_C$  do not commute with the  $\mathcal{N}=(0, 1/2)$  or  $\mathcal{N}=(0, 1)$  supercharges, or generally with the generators of bosonic symmetries realized on the supercharges  $Q_\alpha$  or  $Q_\alpha^i$ . On the other hand, the operators  $P_C$  commute with the covariant spinor derivatives  $D_\alpha, \bar{D}_{\dot{\alpha}}$  or  $D_\alpha^k, \bar{D}_{k\dot{\alpha}}$  defined in the corresponding superspaces. Therefore,  $Q$ -deformations preserve superfield constraints involving these spinor derivatives, in particular the conditions of chirality, antichirality and Grassmann harmonic analyticity. An alternative possibility is the non-anticommutative  $D$ -deformation [7],

defined by a the Poisson operator bilinear in the covariant spinor derivatives. Such deformations preserve the entire supersymmetry but break chirality, which makes it difficult to construct  $D$ -deformed interactions of chiral superfields<sup>3</sup>. As distinct from the  $Q$ -deformations, no stringy interpretation is known for the  $D$ -deformation.

Non-anticommutative  $Q$ -deformations (1.6) or (1.7) differ in a crucial aspect from the bosonic deformations (1.2). Their Poisson operators  $P_C$  are built of mutually anticommuting operators satisfying the nilpotency property  $(Q_\alpha)^3 = 0$  or  $(Q_\alpha^k)^5 = 0$ , respectively. Therefore, the power expansions of the exponentials in (1.6) and (1.7) terminate at corresponding orders. As a result, the ensuing models contain only a *finite* number of local deformation terms in their Lagrangians. In other words, non-anticommutative theories are always local, as opposed to Moyal-deformed theories based on (1.2), which bring an infinite number of new vertices into the Lagrangian.

Mathematically rigorous treatment of the  $\star$ -products for the deformation of both bosonic and Grassmann coordinates in the framework of noncommutative field theory is discussed in [28] using the language of quantum (super)groups and Hopf algebras. In this interpretation, the broken space-time symmetries and supersymmetries of the noncommutative theories are not lost but just deformed. The generators of the deformed (quantum) symmetries by definition act covariantly on the  $\star$ -products of the corresponding fields or superfields, which guarantees the invariance of the action under the deformed (quantum) (super)symmetry transformations. In this review we will not deal with the deformed (quantum) (super)symmetries, since their implications for non-anticommutatively deformed theories are still obscure.

The renormalizability and other quantum aspects of theories with non-anticommutative  $Q$ -deformations of  $\mathcal{N}=(1/2, 1/2)$  supersymmetry were considered in detail in [15]-[20]. Up to now, the case of extended quantum supersymmetry has been studied only for the particular case of chiral singlet  $Q$ -deformations (in the harmonic superspace approach) [32, 33]. In particular, it was found that the non-anticommutative models of the abelian gauge superfield and the neutral hypermultiplet are renormalizable. These results were obtained by computing the divergent contributions to the quantum effective actions. These divergent contributions do not have the form of classical interactions, whence one might conclude that multiplicative renormalizability is jeopardized. Yet, all the divergent terms in the effective action can be removed by a simple field redefinition, viz. by a shift of the scalar field  $\phi$  in the vector gauge multiplet. Since such a field redefinition does not influence the dynamics of the theory and it follows that the divergences in the given case are unphysical. Therefore, the considered theories are not only renormalizable, but actually finite. An analogous situation had been observed in [19] while proving the renormalizability of the  $\mathcal{N}=1/2$  supersymmetric gauge theory. In this case, the divergences are removed by shifting one of the gaugini belonging to the gauge supermultiplet. We remark that in the undeformed limit the actions of the abelian gauge superfield and the neutral hypermultiplet considered in [32] reduce to free ones. This implies that all interactions in these deformed theories are caused by the deformation.

In [33] we also studied the quantum structure of the non-anticommutative charged hypermultiplet model introduced in [29]. This model is of interest because in the undeformed limit it remains interacting, becoming the  $\mathcal{N}=(1, 1)$  supersymmetric extension of electrody-

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<sup>3</sup>The singlet  $D$ -deformation of the  $\mathcal{N}=(1, 1)$  gauge theory was considered in [21, 22]. In this model supersymmetry is preserved, the superfield geometry in the full superspace is deformed, but the Grassmann-analytic representations remain undeformed.

namics. It is well known that the low-energy effective action of the latter model is described by a holomorphic potential which plays an important role in  $\mathcal{N}=2$  Seiberg-Witten theory [5]. In [33], by quantum superfield calculations in harmonic superspace, it was established that this non-anticommutative model is renormalizable in the standard sense. In addition, finite contributions to the low-energy effective action were obtained including the holomorphic potential, which turned out to be deformed in the naive sense. Thus, by now, all abelian models with non-anticommutative chiral singlet  $Q$ -deformation of  $\mathcal{N}=(1,1)$  supersymmetry have been proved renormalizable.

The review is organized as follows. In Sect. 2 we introduce general chiral  $Q$ -deformations of  $\mathcal{N}=(1,1)$  superspace and consider chiral singlet  $Q$ -deformations in harmonic superspace. In Sect. 3 we present superfield formulations of the classical actions for the supersymmetric gauge multiplet and hypermultiplet models with chiral singlet  $Q$ -deformation of  $\mathcal{N}=(1,1)$  supersymmetry in harmonic superspace. In Sect. 4 the component structure of these actions in the abelian case is given. Sect. 5 is devoted to proving renormalizability of the abelian theories of the hypermultiplet and gauge superfield. In Sect. 6 we describe the general structure of the effective action in the charged hypermultiplet model and evaluate the leading (holomorphic) contributions to the effective action. In Sect. 6 we also study the component structure of the new contributions to the low-energy effective action induced by the non-anticommutativity. In the Conclusions the main results are summarized and some further directions are outlined. Two Appendices contain basic relations of Euclidean  $\mathcal{N}=(1,1)$  supersymmetry and Euclidean harmonic superspace.

The review is mostly based on our papers [21, 26, 29, 32, 33]. We shall keep to the notation used in these works.

## 2 Chiral deformations of $\mathcal{N}=(1,1)$ supersymmetry

### 2.1 Chiral deformations of $\mathcal{N}=(1,1)$ superspace

The non-anticommutative chiral deformations are possible only in the Euclidean superspace. Therefore we consider the Euclidean  $\mathcal{N}=(1,1)$  superspace parametrized by the coordinates  $z = (x^m, \theta_i^\alpha, \bar{\theta}^{\dot{\alpha}i})$ , where  $x^m$  are the coordinates of the Euclidean space  $\mathbb{R}^4$  and  $\theta_i^\alpha, \bar{\theta}^{\dot{\alpha}i}$  are Grassmann coordinates. Here  $\alpha, \dot{\alpha} = 1, 2$  denote the spinor indices,  $i = 1, 2$  is the index of the  $R$ -symmetry group  $SU(2)$ . Note that the group  $SO(4)$  of rotations of the Euclidean space  $\mathbb{R}^4$  plays the role similar to the Lorentz group for the Minkowski space  $\mathbb{R}^{3,1}$ . The corresponding universal covering group for  $SO(4)$  is  $Spin(4) = SU(2)_L \times SU(2)_R$ . Therefore the spinors of different chiralities transform independently with respect to the subgroups  $SU(2)_L, SU(2)_R$  and they are not related to each other by the complex conjugation. The basic definitions related to the  $\mathcal{N}=(1,1)$  superspace are collected in the Appendix 1.

It is important to realize that there are two different types of complex conjugation in the  $\mathcal{N}=(1,1)$  superspace [21]. The first one, by definition, acts on the superspace coordinates and superfields as follows

$$\widetilde{\theta}_k^\alpha = \varepsilon^{kj} \varepsilon_{\alpha\beta} \theta_j^\beta, \quad \widetilde{\bar{\theta}}^{\dot{\alpha}k} = -\varepsilon_{kj} \varepsilon_{\dot{\alpha}\dot{\beta}} \bar{\theta}^{\dot{\beta}j}, \quad \widetilde{x^m} = x^m, \quad \widetilde{AB} = \widetilde{B}\widetilde{A}. \quad (2.1)$$

Clearly, the conjugation (2.1) squares to the identity on any object and is compatible with both  $Spin(4)$  and  $R$ -symmetry  $SU(2)$  groups of  $\mathcal{N}=(1,1)$  superspace, preserving the irreducible



representations of these groups. However, this conjugation is incompatible with the reduction of  $\mathcal{N}=(1,1)$  supersymmetry down to  $\mathcal{N}=(1/2,1/2)$  since it is impossible to define the invariant under (2.1) subset of supercharges forming the  $\mathcal{N}=(1/2,1/2)$  supersymmetry <sup>4</sup>.

There is an alternative conjugation in  $\mathcal{N}=(1,1)$  superspace denoted by “\*” and defined by the rules

$$(\theta_k^\alpha)^* = \varepsilon_{\alpha\beta} \theta_k^\beta, \quad (\bar{\theta}^{\dot{\alpha}k})^* = \varepsilon_{\dot{\alpha}\dot{\beta}} \bar{\theta}^{\dot{\beta}k}, \quad (x^m)^* = x^m, \quad (AB)^* = B^* A^*. \quad (2.2)$$

The conjugation (2.2) is compatible with the reduction of the  $\mathcal{N}=(1,1)$  supersymmetry down to  $\mathcal{N}=(1/2,1/2)$  since it allows to single out the invariant  $\mathcal{N}=(1/2,1/2)$  subspaces in the  $\mathcal{N}=(1,1)$  superspace. It respects also the action of the group  $Spin(4)$ . However, this conjugation squares to the identity only on the bosonic coordinates and fields, while for the spinor fields the double conjugation yields  $-1$ . Therefore it is natural to refer to the involution (2.2) as a “pseudoconjugation”. The action of the  $R$ -symmetry group  $SU(2)$  on  $\mathcal{N}=(1,1)$  supercharges and fermionic coordinates of  $\mathcal{N}=(1,1)$  superspace is incompatible with the pseudoconjugation (2.2), while it preserves the  $SL(2,R)$  group which plays the role of  $R$ -symmetry group in this case. This means that the pseudoconjugation \* corresponds to another real form of  $\mathcal{N}=(1,1)$  supersymmetry with a non-compact group of internal automorphisms. The undeformed real superfield actions in these two different Euclidean  $\mathcal{N}=(1,1)$  superspaces are related to each other and to  $\mathcal{N}=2$  supersymmetric actions in Minkowski space by the Wick rotations. It should be pointed out that, when the deformations of supersymmetry (or other symmetries) are introduced, the actions which are real with respect to one conjugation can be complex with respect to the other, and vice versa. In what follows we shall deal with only one type of the conjugation, that given by eq. (2.1).

The chiral deformations of supersymmetry appear most naturally in the chiral coordinates,

$$z_L = (x_L^m, \theta_i^\alpha, \bar{\theta}^{\dot{\alpha}k}), \quad x_L^m = x^m + i(\sigma^m)_{\alpha\dot{\alpha}} \theta_k^\alpha \bar{\theta}^{\dot{\alpha}k}, \quad (2.3)$$

where the Euclidean sigma-matrices are given in the Appendix 1, (A.3). The supertranslations act on the coordinates  $z_L$  as follows

$$\delta_\epsilon x_L^m = 2i(\sigma^m)_{\alpha\dot{\alpha}} \theta_k^\alpha \bar{\epsilon}^{\dot{\alpha}k}, \quad \delta_\epsilon \theta_k^\alpha = \epsilon_k^\alpha, \quad \delta_\epsilon \bar{\theta}^{\dot{\alpha}k} = \bar{\epsilon}^{\dot{\alpha}k}, \quad (2.4)$$

where  $\epsilon_k^\alpha, \bar{\epsilon}^{\dot{\alpha}k}$  are anticommuting parameters. In the chiral coordinates, the supercharges and covariant spinor derivatives (A.4) read

$$Q_\alpha^i = \partial_\alpha^i, \quad \bar{Q}_{\dot{\alpha}i} = -\bar{\partial}_{\dot{\alpha}i} + 2i\theta_i^\alpha (\sigma_m)_{\alpha\dot{\alpha}} \frac{\partial}{\partial x_L^m}, \quad (2.5)$$

$$D_\alpha^i = \partial_\alpha^i + 2i\bar{\theta}^{\dot{\alpha}i} (\sigma_m)_{\alpha\dot{\alpha}} \frac{\partial}{\partial x_L^m}, \quad \bar{D}_{\dot{\alpha}i} = -\bar{\partial}_{\dot{\alpha}i}. \quad (2.6)$$

Consider now the operator  $P_C$  defined in the coordinate basis (2.3) by the following expression

$$P_C = -\overleftarrow{\partial}_\alpha^i C_{ij}^{\alpha\beta} \overrightarrow{\partial}_\beta^j = -\overleftarrow{Q}_\alpha^i C_{ij}^{\alpha\beta} \overrightarrow{Q}_\beta^j. \quad (2.7)$$

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<sup>4</sup>Respectively, in  $\mathcal{N}=(1,1)$  superspace with the conjugation (2.1) there are no subspaces closed under the  $\mathcal{N}=(1/2,1/2)$  supersymmetry.

It acts on the arbitrary superfields  $A, B$  according to the rules

$$\begin{aligned} AP_C B &= -(-1)^{p(A)} (\partial_\alpha^i A) C_{ij}^{\alpha\beta} (\partial_\beta^j B), \\ AP_C^2 B &= (\partial_\gamma^k \partial_\alpha^i A) C_{ij}^{\alpha\beta} C_{kl}^{\gamma\rho} (\partial_\rho^l \partial_\beta^j B). \end{aligned} \quad (2.8)$$

Here  $C_{ij}^{\alpha\beta}$  are some constants and  $p(A)$  is the Grassmann parity of the superfield  $A$ . The operator  $P_C$  defines the Moyal-Weyl  $\star$ -product of superfields (see (1.7)),

$$A \star B = A e^{P_C} B = AB + AP_C B + \frac{1}{2} AP_C^2 B + \frac{1}{6} AP_C^3 B + \frac{1}{24} AP_C^4 B. \quad (2.9)$$

The operator  $P_C$  is nilpotent since  $(\partial_\alpha^i)^5 = 0$ . Therefore the  $\star$ -deformation (2.9) never produces non-localities, in contrast to the deformations of bosonic coordinates (1.1) (see, e.g., [9]).

As the operator (2.7) is built out only of the supercharges, and they anticommute with the covariant derivatives (2.6), the product (2.9) preserves both chirality and antichirality,

$$\begin{aligned} D_\alpha^i (A \star B) &= (D_\alpha^i A) \star B + A \star (D_\alpha^i B), \\ \bar{D}_{i\dot{\alpha}} (A \star B) &= (\bar{D}_{i\dot{\alpha}} A) \star B + A \star (\bar{D}_{i\dot{\alpha}} B). \end{aligned} \quad (2.10)$$

What is more important for  $\mathcal{N}=(1,1)$  supersymmetric theories, the  $\star$ -multiplication also respects the Grassmann harmonic analyticity (see the next subsection for details). Since all  $\mathcal{N}=(1,1)$  supersymmetric Euclidean theories are well defined only if the chirality and harmonic analyticity are preserved (similarly to  $\mathcal{N}=2$  models in Minkowski space), it is a consistent deformation of these theories when the standard multiplication in their classical actions is replaced by the  $\star$ -product (2.9).

It is natural to demand the multiplication (2.9) to be consistent with the reality properties. Since in the Euclidean superspaces there are two different conjugations (2.1) and (2.2) which respect either  $SU(2)$  or  $SL(2, R)$   $R$ -symmetry groups, the preservation of reality puts two different constraints on the parameters of deformations  $C_{ij}^{\alpha\beta}$ :

$$\widetilde{(A \star B)} = \tilde{B} \star \tilde{A} \quad \implies \quad \widetilde{C_{ij}^{\alpha\beta}} = C_{\alpha\beta}^{ij}, \quad (2.11)$$

$$(A \star B)^* = B^* \star A^* \quad \implies \quad (C_{ij}^{\alpha\beta})^* = C_{\alpha\beta ij}. \quad (2.12)$$

In our further consideration we restrict ourselves to the case of the conjugation (2.11).

In general, since the constants  $C_{ij}^{\alpha\beta}$  have both the spinor and  $R$ -symmetry group indices, the Euclidean  $SO(4)$  and  $SU(2)_L$  groups, as well as the  $R$ -symmetry group  $SU(2)$ , are broken in the theories with the deformations induced by the  $\star$ -product (2.9). Moreover, the  $\mathcal{N}=(1,1)$  supersymmetry is also broken down to  $\mathcal{N}=(1,0)$  since the product (2.9) involves only the  $\mathcal{N}=(1,0)$  supercharges  $Q_\alpha^i$  which have non-vanishing anticommutators with the  $\mathcal{N}=(0,1)$  supercharges  $\bar{Q}_{i\dot{\alpha}}$ .

Taking into account the definition (2.7), the  $\star$ -multiplication (2.9) of two superfields can always be written as

$$A \star B = AB + Q_\alpha^k N_k^\alpha(A, B), \quad (2.13)$$

where  $N_k^\alpha(A, B)$  is some function of the superfields  $A, B$  and constants  $C_{ij}^{\alpha\beta}$ . Eq. (2.13) implies that in the full superspace integral the  $\star$ -product of two superfields reduces to the usual product,

$$\int d^4 x_L d^4 \theta d^4 \bar{\theta} A \star B = \int d^4 x_L d^4 \theta d^4 \bar{\theta} AB. \quad (2.14)$$

In a similar way one can check that under the superspace integral the  $\star$ -product of three superfields obeys the cyclic property

$$\int d^8z A \star B \star C = \int d^8z C \star A \star B. \quad (2.15)$$

There is also an analog of the relation (2.14) for the chiral subspace,

$$\int d^4x_L d^4\theta A \star B = \int d^4x_L d^4\theta A B. \quad (2.16)$$

The relation (2.16) is formally valid not only for the chiral superfields, but also for general ones  $A, B$  (i.e. those given on the full  $\mathcal{N}=(1,1)$  superspace). However, this is not case for the general  $\mathcal{N}=(1,1)$  superfields under the antichiral integral,

$$\int d^4x_R d^4\bar{\theta} A \star B \neq \int d^4x_R d^4\bar{\theta} A B. \quad (2.17)$$

Only for the antichiral superfields  $\bar{\Phi}, \bar{\Lambda}$  the equality sign in (2.17) is restored,

$$\int d^4x_R d^4\bar{\theta} \bar{\Phi} \star \bar{\Lambda} = \int d^4x_R d^4\bar{\theta} \bar{\Phi} \bar{\Lambda}. \quad (2.18)$$

Note that in the antichiral coordinates one should use the following expressions for supercharges and covariant spinor derivatives

$$Q_\alpha^i = \partial_\alpha^i - 2i\bar{\theta}^{\dot{\alpha}i}(\sigma_m)_{\alpha\dot{\alpha}} \frac{\partial}{\partial x_R^m}, \quad \bar{Q}_{\dot{\alpha}i} = -\bar{\partial}_{\dot{\alpha}i}, \quad (2.19)$$

$$D_\alpha^i = \partial_\alpha^i, \quad \bar{D}_{\dot{\alpha}i} = -\bar{\partial}_{\dot{\alpha}i} - 2i\theta_i^\alpha(\sigma_m)_{\alpha\dot{\alpha}} \frac{\partial}{\partial x_R^m}, \quad (2.20)$$

where  $x_R^m = x^m - i(\sigma^m)_{\alpha\dot{\alpha}}\theta^\alpha\bar{\theta}^{\dot{\alpha}k}$ .

Now, let us define the  $\star$ -commutators and anticommutators of operators and superfields as follows

$$[A \star B] = A \star B - B \star A, \quad \{A \star B\} = A \star B + B \star A. \quad (2.21)$$

It is instructive to find the  $\star$ -(anti)commutators of the bosonic and fermionic superspace coordinates,

$$\begin{aligned} \{\theta_k^\alpha \star \theta_j^\beta\} &= 2C_{ij}^{\alpha\beta}, & [x_L^m \star x_L^n] &= 0, & [x_L^m \star \theta_k^\alpha] &= 0, \\ [x_L^m \star \bar{\theta}^{\dot{\alpha}k}] &= 0, & \{\theta_k^\alpha \star \bar{\theta}^{\dot{\beta}j}\} &= 0, & \{\bar{\theta}^{\dot{\alpha}k} \star \bar{\theta}^{\dot{\beta}j}\} &= 0. \end{aligned} \quad (2.22)$$

Eqs. (2.22) tell us that in the chiral basis the  $\star$ -product affects only the anticommutator of left-chiral coordinates  $\theta_i^\alpha$ .

The constant tensor  $C_{ij}^{\alpha\beta}$  can be decomposed into the traceless part and trace with respect to the  $SU(2)_L$  spinor and  $SU(2)$   $R$ -symmetry indices,

$$C_{ij}^{\alpha\beta} = C_{(ij)}^{(\alpha\beta)} + \varepsilon^{\alpha\beta} \varepsilon_{ij} I. \quad (2.23)$$

The Poisson operator (2.7) acquires the most simple form in the particular case  $C_{(ij)}^{(\alpha\beta)} = 0$ :

$$P_s = -\overleftarrow{Q}_\alpha^i I \varepsilon^{\alpha\beta} \varepsilon_{ij} \overrightarrow{Q}_\beta^j = -\overleftarrow{Q}_\alpha^i I \overrightarrow{Q}_i^\alpha = -\overleftarrow{\partial}_\alpha^i I \overrightarrow{\partial}_i^\alpha. \quad (2.24)$$

The operator  $P_s$  produces the following  $\star$ -product

$$A \star B = A e^{P_s} B. \quad (2.25)$$

Clearly, the deformation (2.25) does not break the symmetries with respect to the Euclidean rotation group  $SO(4)$  and the  $R$ -symmetry group  $SU(2)$ . However, in the deformed theories corresponding to the operator  $P_s$ ,  $\mathcal{N}=(1,1)$  supersymmetry is still broken by half.

The non-anticommutative  $Q$ -deformation associated with the  $\star$ -product (2.25) and preserving the maximal number of symmetries will be referred to as the chiral singlet deformation. In what follows we will consider only this type of deformations because of its uniqueness and relative simplicity.

## 2.2 Chiral singlet deformation of $\mathcal{N}=(1,1)$ harmonic superspace

$\mathcal{N}=2$ ,  $D=4$  harmonic superspace (its Minkowski space version) was pioneered in [34]. The pedagogical introduction to the harmonic superspace approach can be found in the book [35]. Here, following ref. [21], we present how the non-anticommutative deformations given by the operator (2.24) are realized in harmonic superspace. The salient features of the Euclidean version of harmonic superspace are collected in the Appendix 2.

The complex conjugation (2.1) can be naturally extended to the harmonic variables,

$$\widetilde{u_k^\pm} = u^{\pm k}. \quad (2.26)$$

Using (2.1) and (2.26) one can find the complex conjugation rules for the harmonic superspace coordinates  $x_A^m$ ,  $\theta^{\pm\alpha}$ ,  $\bar{\theta}^{\pm\dot{\alpha}}$

$$\widetilde{x_A^m} = x_A^m, \quad \widetilde{\theta^{\pm\alpha}} = \varepsilon_{\alpha\beta} \theta^{\pm\beta}, \quad \widetilde{\bar{\theta}^{\pm\dot{\alpha}}} = \varepsilon_{\dot{\alpha}\dot{\beta}} \bar{\theta}^{\pm\dot{\beta}}. \quad (2.27)$$

Note that the involution  $\sim$  is a pseudoconjugation since it squares to  $-1$  while acting on the harmonics and harmonic projections of Grassmann coordinates.

Let us now apply to the Poisson operator  $P_s$  (2.24) of the chiral singlet deformations with the  $\star$ -product (2.25). In harmonic superspace, this operator can be written as

$$P_s = I(\overleftarrow{Q}^{+\alpha} \overrightarrow{Q}_\alpha^- - \overleftarrow{Q}^{-\alpha} \overrightarrow{Q}_\alpha^+), \quad (2.28)$$

where  $Q_\alpha^\pm = Q_\alpha^i u_i^\pm$  are the harmonic projections of supercharges. In terms of the supercharges  $Q_\alpha^\pm$  the  $\star$ -product (2.25) is rewritten as

$$\star = e^{P_s} = 1 + P_s + \frac{1}{2} P_s^2 + \frac{1}{6} P_s^3 + \frac{1}{24} P_s^4, \quad (2.29)$$

where

$$\begin{aligned} \frac{1}{2} P_s^2 &= -\frac{I^2}{4} [(\overleftarrow{Q}^+)^2 (\overrightarrow{Q}^-)^2 + (\overleftarrow{Q}^-)^2 (\overrightarrow{Q}^+)^2] - I^2 \overleftarrow{Q}^{+\alpha} \overleftarrow{Q}^{-\beta} \overrightarrow{Q}_\alpha^- \overrightarrow{Q}_\beta^+, \\ \frac{1}{6} (P_s)^3 &= -\frac{I^3}{3} [(\overleftarrow{Q}^-)^2 \overleftarrow{Q}^{+\alpha} (\overrightarrow{Q}^+)^2 \overrightarrow{Q}_\alpha^- - (\overleftarrow{Q}^+)^2 \overleftarrow{Q}^{-\alpha} (\overrightarrow{Q}^-)^2 \overrightarrow{Q}_\alpha^+], \\ \frac{1}{24} (P_s)^4 &= \frac{I^4}{16} (\overleftarrow{Q}^-)^2 (\overleftarrow{Q}^+)^2 (\overrightarrow{Q}^+)^2 (\overrightarrow{Q}^-)^2. \end{aligned} \quad (2.30)$$

Note that  $P_s$  commutes with the spinor derivatives in the analytic basis (they are defined in (A.12)),

$$[P_s, D_\alpha^\pm] = 0, \quad [P_s, \bar{D}_{\dot{\alpha}}^\pm] = 0. \quad (2.31)$$

The property (2.31) shows that the  $\star$ -product (2.29) preserves the harmonic Grassmann analyticity. In other words, the  $\star$ -product of two analytic superfields  $\Phi_A, \Psi_A$  is again an analytic superfield,

$$(D_\alpha^\pm, \bar{D}_{\dot{\alpha}}^\pm)(\Phi_A \star \Psi_A) = 0. \quad (2.32)$$

Due to the simple relation between the supercharges and covariant spinor derivatives

$$Q_\alpha^\pm = D_\alpha^\pm + 2i\bar{\theta}^{\pm\dot{\alpha}}(\sigma_m)_{\alpha\dot{\alpha}}\partial_m, \quad (2.33)$$

the following relations are valid for an arbitrary analytic superfield  $\Phi_A$

$$Q_\alpha^+\Phi_A = 2i\bar{\theta}^{+\dot{\alpha}}(\sigma_m)_{\alpha\dot{\alpha}}\partial_m\Phi_A, \quad (Q^+)^2\Phi_A = 4(\bar{\theta}^+)^2\Box\Phi_A. \quad (2.34)$$

The equations (2.34) imply that in the decomposition of the  $\star$ -product (2.29) any term involving more than two  $Q_\alpha^+$  supercharges on the analytic superfields vanishes, e.g.,

$$(Q^+)^2\Phi_A Q_\alpha^+\Psi_A = 4i(\bar{\theta}^+)^2\Box\Phi_A\bar{\theta}^{+\dot{\alpha}}(\sigma_m)_{\alpha\dot{\alpha}}\partial_m\Psi_A = 0. \quad (2.35)$$

As a consequence, the singlet  $\star$ -product of two analytic superfields is at most quadratic in the deformation parameter  $I$ :

$$\begin{aligned} \Phi_A \star \Psi_A &= \Phi_A \Psi_A + I(-1)^{p(\Phi)}(Q^{+\alpha}\Phi_A Q_\alpha^-\Psi_A - Q^{-\alpha}\Phi_A Q_\alpha^+\Psi_A) \\ &\quad - \frac{I^2}{4}[(Q^+)^2\Phi_A(Q^-)^2\Psi_A + (Q^-)^2\Phi_A(Q^+)^2\Psi_A] - I^2 Q^{+\alpha}Q^{-\beta}\Phi_A Q_\alpha^- Q_\beta^+\Psi_A. \end{aligned} \quad (2.36)$$

Then it is easy to see that the  $\star$ -commutator of analytic superfields is linear in  $I$

$$\begin{aligned} [\Phi_A \star \Psi_A] &= \Phi_A P_s \Psi_A - \Psi_A P_s \Phi_A = 2\Phi_A P_s \Psi_A \\ &= 2I(Q^{+\alpha}\Phi_A Q_\alpha^-\Psi_A - Q^{-\alpha}\Phi_A Q_\alpha^+\Psi_A). \end{aligned} \quad (2.37)$$

The operator of chiral singlet deformations (2.24) also commutes with the harmonic derivatives (A.13),

$$[P_s, D^{++}] = 0, \quad [P_s, D^{--}] = 0. \quad (2.38)$$

As a result, the chiral singlet deformation does not break the internal symmetry group  $SU(2)$  represented by the harmonics  $u_i^\pm$ . It also preserves the Grassmann shortness conditions,  $D^{\pm\pm}\Phi = 0$ . This makes it possible to utilize short multiplets while constructing the actions, like in the undeformed theories.

The properties of the chiral singlet deformation listed above (the preservations of left and right chiralities, as well as of the Grassmann analyticity and Grassmann shortness) indicate that the the harmonic superspace approach is equally applicable to the  $\mathcal{N}=(1,1)$  non-anticommutative superfield theories, as to the conventional  $\mathcal{N}=2$  supersymmetric ones.

### 3 Classical non-anticommutative models in $\mathcal{N}=(1,1)$ harmonic superspace

In constructing classical superfield actions of non-anticommutative theories we follow the simple rule: in order to obtain the action of a non-anticommutative model one should replace the usual product of superfields in the action of the corresponding undeformed model by the  $\star$ -product (2.25).

#### 3.1 Super-Yang-Mills model

In the harmonic superspace approach [35] the gauge multiplet of  $\mathcal{N}=2$  or  $\mathcal{N}=(1,1)$  supersymmetry is described by the analytic superfield  $V^{++}$  with the harmonic  $U(1)$  charge  $+2$ . In general, this superfield is valued in the Lie algebra of the gauge group  $U(n)$ , i.e., it can be written as  $V^{++} = V^{++M} T^M$ , where  $T^M$  are the generators of  $U(n)$ .

Under the deformed  $U(n)$  gauge group the gauge superfield is assumed to transform as

$$\delta_\Lambda V^{++} = D^{++}\Lambda + [V^{++} \star \Lambda], \quad (3.1)$$

where  $\Lambda$  is an analytic superfield parameter also taking values in the algebra of the gauge group. Note that even in the  $U(1)$  case, i.e. with only one copy of  $V^{++}$  and  $\Lambda$ , the transformation rule (3.1) is still non-abelian due to the presence of the  $\star$  product in the second term. In the undeformed limit this “non-abelian” piece vanishes. In the case of “genuine” non-abelian  $\mathcal{N}=(1,1)$  gauge theory there are two sources of the non-abelian structure, the standard one surviving in the undeformed limit and the one induced by the  $\star$ -product.

To construct an action which is invariant under the gauge transformations (3.1) representing the deformed gauge  $U(n)$  group we follow the same steps as in the non-abelian  $\mathcal{N}=2$  super Yang-Mills theory in harmonic superspace [34, 35]. We introduce the superfield  $V^{--}$  as a solution of the harmonic zero-curvature equation,

$$D^{++}V^{--} - D^{--}V^{++} + [V^{++} \star V^{--}] = 0. \quad (3.2)$$

The solution of (3.2) is given by the following series [36]

$$V^{--}(z, u) = \sum_{n=1}^{\infty} (-1)^n \int du_1 \dots du_n \frac{V^{++}(z, u_1) \star V^{++}(z, u_2) \star \dots \star V^{++}(z, u_n)}{(u^+ u_1^+)(u_1^+ u_2^+) \dots (u_n^+ u^+)}, \quad (3.3)$$

where  $(u_1^+ u_2^+)^{-1}$  is a harmonic distribution introduced in [34]. Using the superfield  $V^{--}$ , one can construct the gauge superfield strengths in the standard manner,

$$W = -\frac{1}{4}(\bar{D}^+)^2 V^{--}, \quad \bar{W} = -\frac{1}{4}(D^+)^2 V^{--}. \quad (3.4)$$

As in the usual  $\mathcal{N}=2$  SYM theory, these superfields satisfy the Bianchi identity  $(D^+)^2 W = (\bar{D}^+)^2 \bar{W}$ . Applying the relations (3.1), (3.2) and the gauge transformation rule for the  $V^{--}$  prepotential,  $\delta_\Lambda V^{--} = D^{--}\Lambda + [V^{--} \star \Lambda]$ , it is easy to show that the superfield strengths (3.4) transform covariantly under the gauge group,

$$\delta_\Lambda W = [W \star \Lambda], \quad \delta_\Lambda \bar{W} = [\bar{W} \star \Lambda]. \quad (3.5)$$

Moreover, they are covariantly (anti)chiral

$$\begin{aligned}\bar{D}_\alpha^+ W &= 0, & \bar{D}_\alpha^- W - [\bar{D}_\alpha^+ V^{--} \star W] &= 0, \\ D_\alpha^+ \bar{W} &= 0, & D_\alpha^- \bar{W} - [D_\alpha^+ V^{--} \star \bar{W}] &= 0,\end{aligned}\tag{3.6}$$

and are covariantly independent of harmonics,

$$D^{++}W + [V^{++} \star W] = 0, \quad D^{++}\bar{W} + [V^{++} \star \bar{W}] = 0.\tag{3.7}$$

The equations (3.1)–(3.7) have exactly the same form as in the corresponding undeformed non-abelian  $\mathcal{N}=2$  super Yang-Mills theory. Therefore, the classical action of the non-anticommutative supersymmetric gauge theory can be also represented as an integral over the chiral subspace

$$S_{SYM} = \frac{1}{4} \text{tr} \int d^4 x_L d^4 \theta W^2.\tag{3.8}$$

Note that, due to the property (2.14), the  $\star$ -product of two superfield strengths in (3.8) is reduced to the ordinary product. However, despite the absence of the  $\star$ -product in (3.8), the non-anticommutative deformation is still present in this expression through the superfield strengths (3.4) and the prepotential  $V^{--}$  (3.3). It is easy to check that the action (3.8) is gauge invariant,

$$\delta_\Lambda S_{SYM} = \frac{1}{4} \text{tr} \int d^4 x_L d^4 \theta [W^2 \star \Lambda] = 0.\tag{3.9}$$

Here we have applied the equations (2.14), (3.5). One can also show that this action does not depend on the harmonic and Grassmann variables,

$$D^{++}S_{SYM} = 0, \quad \bar{D}_\alpha^\pm S_{SYM} = 0.\tag{3.10}$$

It should also be noted that the chiral action (3.8) is real in the Euclidean case.

The classical action of non-anticommutative supersymmetric gauge theory can be expressed as a full superspace integral of the Lagrangian written in terms of the analytic superfields  $V^{++}$ , quite analogously to the action of the usual non-abelian  $\mathcal{N}=2$  gauge theory [36],

$$S_{SYM}[V^{++}] = \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \text{tr} \int d^{12} z du_1 \dots du_n \frac{V^{++}(z, u_1) \star V^{++}(z, u_2) \star \dots \star V^{++}(z, u_n)}{(u_1^+ u_2^+)(u_2^+ u_3^+) \dots (u_n^+ u_1^+)}.\tag{3.11}$$

While passing to the quantum theory, the representation (3.11) for the classical action proves to be more advantageous.

## 3.2 Hypermultiplet model

The hypermultiplet in harmonic superspace is described either by a complex analytic superfield  $q^+$  with the  $U(1)$  charge  $+1$  or by a real analytic chargeless superfield  $\omega$ . Both these descriptions are known to be related to each other via some sort of duality [35]. Therefore we can confine our consideration to the  $q$ -hypermultiplet models.

The free classical action of the  $q^+$  superfield in harmonic superspace is given by

$$S_0[q^+] = - \int d\zeta du \tilde{q}^+ D^{++} q^+.\tag{3.12}$$

Here  $\tilde{q}^+$  is a superfield conjugated to  $q^+$  and  $d\zeta du = d^4x_A d^4\theta^- du$  is the integration measure of the analytic superspace. The rules of integration in harmonic superspace are given in the Appendix 2, eq. (A.14). Note that the identity (2.14) allows us to omit the  $\star$ -product in the free superfield actions like (3.12). In other words, the chiral singlet deformation does not modify the free actions and affects only the interaction terms. We will show that both the hypermultiplet self-interaction and the interaction of hypermultiplet with a vector multiplet are deformed due to non-anticommutativity. In some special cases considered below this interaction disappears when the deformation is turned off.

It is easy to write the quartic interaction term of the  $q$ -superfields [21],

$$S_4[q^+] = \int d\zeta du (a \tilde{q}^+ \star q^+ \star \tilde{q}^+ \star q^+ + b q^+ \star q^+ \star \tilde{q}^+ \star \tilde{q}^+), \quad (3.13)$$

where  $a, b$  are coupling constants. Note that two terms in the action (3.13) differ only by ordering of superfields with respect to the  $\star$ -product. In the undeformed limit  $I \rightarrow 0$  both these terms are reduced to the single standard interaction term  $(\tilde{q}^+ q^+)^2$ , with the coupling constant  $a + b$ .

Let us now introduce the interaction of hypermultiplet with the background gauge superfields.

As is well known, the interaction of matter fields with the gauge ones is to large extent specified by the choice of the representation of the gauge group to which matter fields belong. In particular, the fundamental and adjoint representations are of the main interest in quantum field theory.

Let us start with the fundamental representation. In this case the superfield  $V^{++}$  is a matrix which belongs to the Lie algebra of the gauge group  $U(n)$  acting on the complex  $n$ -plet of superfields  $q^+$ .

Based on the analogy with the ordinary  $U(n)$  gauge theory, the model (3.12) can be coupled to the gauge superfield in the standard way, i.e. just by replacing the flat harmonic derivative  $D^{++}$  with the corresponding covariant one  $\nabla^{++} = D^{++} + V^{++}\star$ . As a result, the action of the non-anticommutative hypermultiplet superfield interacting with the vector superfield in the fundamental representation of the deformed  $U(n)$  group is given by

$$S_f[q^+, V^{++}] = - \int d\zeta du \tilde{q}^+ \star (D^{++} + V^{++}) \star q^+. \quad (3.14)$$

Here  $\tilde{q}^+$  is conjugated to  $q^+$ . It is easy to check that the action (3.14) is invariant under the gauge transformations of vector superfield (3.1) supplemented by the following hypermultiplet transformations

$$\delta_\Lambda \tilde{q}^+ = \tilde{q}^+ \star \Lambda, \quad \delta_\Lambda q^+ = \Lambda \star q^+. \quad (3.15)$$

We refer to the model (3.14) as a non-anticommutative model of charged hypermultiplet [29]. It should be emphasized that the transformation laws (3.15) are essentially non-abelian (they possess a non-zero Lie bracket) even in the  $U(1)$  case. They become the standard  $U(1)$  transformations only in the undeformed limit, when the  $\star$ -product turns into the ordinary one.

In the adjoint representation the hypermultiplet superfield is transformed on pattern of the second term in the transformation law (3.1)

$$\delta_\Lambda \tilde{q}^+ = [\tilde{q}^+ \star \Lambda], \quad \delta_\Lambda q^+ = [q^+ \star \Lambda]. \quad (3.16)$$



Here  $q^+$  is a matrix in the Lie algebra of the gauge group, and it can be expanded over the generators of the gauge group as  $q^+ = q^{+M} T^M$ . The corresponding classical action is given by

$$S_{ad}[q^+, V^{++}] = -\text{tr} \int d\zeta du \tilde{q}^+ \star (D^{++} q^+ + [V^{++} \star q^+]). \quad (3.17)$$

It is easy to check that (3.17) is invariant under the gauge transformations (3.1) supplemented by (3.16).

We refer to the model with the classical action (3.17) and deformed gauge group  $U(1)$  as the non-anticommutative model of neutral hypermultiplet. It is worth noting that in the case of  $U(1)$  gauge group the interaction with the gauge superfield in (3.17) is only due to the non-anticommutative deformation. This interaction disappears in the limit  $I \rightarrow 0$  and the model (3.17) becomes free. This is a new feature specific only for the non-anticommutative neutral hypermultiplet model with the  $U(1)$  gauge group. The interaction still survives in the limit  $I \rightarrow 0$  for the non-abelian neutral hypermultiplet or for the charged hypermultiplet (even with the  $U(1)$  gauge group). In our further consideration we restrict ourselves only to the models with deformed  $U(1)$  gauge group.

It is instructive to rewrite the actions (3.14), (3.17) in a unified form. For this purpose we combine the hypermultiplet superfields  $\tilde{q}^+$ ,  $q^+$  into a single  $SU(2)$  doublet  $q^{+a}$ ,

$$q^{+a} = \varepsilon^{ab} q_b^+ = (\tilde{q}^+, q^+) = \widetilde{q_a^+}, \quad a = 1, 2. \quad (3.18)$$

The covariant harmonic derivative  $\nabla^{++}$  acts on the doublet  $q^{+a}$  in a different way for the adjoint and fundamental representations of the  $U(1)$  gauge group,

$$\text{Adj. rep.:} \quad \nabla^{++} q^{+a} = D^{++} q^{+a} + [V^{++} \star q^{+a}], \quad (3.19)$$

$$\text{Fund. rep.:} \quad \nabla^{++} q^{+a} = D^{++} q^{+a} + \frac{1}{2} [V^{++} \star q^{+a}] - \frac{1}{2} (\tau_3)_b^a \{V^{++} \star q^{+b}\}. \quad (3.20)$$

Here  $\tau_3 = \text{diag}(1, -1)$  is the Pauli matrix. According to the definition (3.19), the expression  $\nabla^{++} q^{+a}$  is covariant with respect to the additional symmetry group  $SU(2)_{PG}$  which is called the Pauli-Gürsey group [35]. The matrices of this group act on the index  $a$  of  $\nabla^{++} q^{+a}$ . Using the new notation, the actions (3.14), (3.17) can be uniformly written as

$$S[q^+, V^{++}] = \frac{1}{2} \int d\zeta du q_a^+ \nabla^{++} q^{+a}. \quad (3.21)$$

In the case of fundamental representation, the symmetry group  $SU(2)_{PG}$  is broken down to  $U(1)$  with the generator  $\tau_3$ .

## 4 The component structure of $\mathcal{N}=(1, 0)$ non-anticommutative abelian models

In the previous section we have shown that in the superfield Lagrangians the chiral singlet deformation leads to some new interaction terms induced by the  $\star$ -product. It is important that this new interaction is always local owing to the nilpotency of the operator  $P_s$ . Here we study these new interaction terms at the component level. The most important features of such Lagrangians can be most clearly exhibited on the examples of abelian models of gauge superfield and hypermultiplet.

## 4.1 Gauge superfield model

The gauge multiplet of  $\mathcal{N}=(1,1)$  supersymmetry consists of two independent real scalar fields  $\phi, \bar{\phi}$ , independent Weyl spinors  $\Psi_\alpha^k, \bar{\Psi}^{\dot{\alpha}k}$  with the internal symmetry group index  $k = 1, 2$  and a triplet of auxiliary fields  $\mathcal{D}^{(kl)}$ . The component structure of the  $\mathcal{N}=(1,0)$  non-anticommutative abelian supergauge model in terms of these fields was studied in [26, 31].

The classical action of non-anticommutative super Yang-Mills model is given by (3.8). In the abelian case we can omit the trace in (3.8),

$$S_{SYM} = \frac{1}{4} \int d^4x_L d^4\theta W^2. \quad (4.1)$$

Note that, according to (3.6), the superfield  $W$  is covariantly chiral rather than manifestly chiral. Therefore it depends on the variables  $\bar{\theta}_\alpha^+$ :

$$W = \mathcal{A} + \bar{\theta}_\alpha^+ \tau^{-\dot{\alpha}} + (\bar{\theta}^+)^2 \tau^{-2}, \quad (4.2)$$

where  $\mathcal{A}, \tau^{-\dot{\alpha}}, \tau^{-2}$  are some chiral superfields. Remarkably, among these superfields only  $\mathcal{A}$  contributes to the action (4.1). Indeed, the relations (3.6) and (3.7) show that the terms involving the superfields  $\tau^{-\dot{\alpha}}, \tau^{-2}$  are always proportional to some  $\star$ -commutators of superfields and therefore vanish under the integral over  $d^4\theta$ . As a result, the action (4.1) acquires the following form

$$S_{SYM} = \frac{1}{4} \int d^4x_L d^4\theta \mathcal{A}^2. \quad (4.3)$$

Let us find the component structure of the superfield  $\mathcal{A}$ . For this purpose we have to fix the component structure of the gauge superfield  $V^{++}$ . Using the gauge freedom (3.1) one can eliminate the lowest components of  $V^{++}$  by effecting the Wess-Zumino gauge,

$$\begin{aligned} V_{WZ}^{++}(x_A^m, \theta^{+\alpha}, \bar{\theta}^{+\dot{\alpha}}, u) &= (\theta^+)^2 \bar{\phi}(x_A) + (\bar{\theta}^+)^2 \phi(x_A) + 2(\theta^+ \sigma_m \bar{\theta}^+) A_m(x_A) + 4(\bar{\theta}^+)^2 \theta^+ \Psi^-(x_A) \\ &\quad + 4(\theta^+)^2 \bar{\theta}^+ \bar{\Psi}^-(x_A) + 3(\theta^+)^2 (\bar{\theta}^+)^2 \mathcal{D}^{--}(x_A), \end{aligned} \quad (4.4)$$

where

$$\Psi_\alpha^-(x_A) = \Psi_\alpha^k(x_A) u_k^-, \quad \bar{\Psi}^{\dot{\alpha}-}(x_A) = \bar{\Psi}^{\dot{\alpha}k}(x_A) u_k^-, \quad \mathcal{D}^{--} = \mathcal{D}^{kl}(x_A) u_k^- u_l^-. \quad (4.5)$$

The residual gauge transformation of the superfield (4.4) reads

$$\delta_r V_{WZ}^{++} = D^{++} \Lambda_r + [V^{++} \star \Lambda_r], \quad \Lambda_r = i\lambda(x_A), \quad (4.6)$$

where  $\lambda(x_A)$  is an arbitrary real function. The transformation (4.6) amounts to the following gauge transformations for the component fields

$$\begin{aligned} \delta\phi &= -8I A_m \partial_m \lambda, & \delta\bar{\phi} &= 0, \\ \delta\Psi_\alpha^k &= -4I (\sigma_m \bar{\Psi}^k)_\alpha \partial_m \lambda, & \delta\bar{\Psi}_\alpha^k &= 0, \\ \delta A_m &= (1 + 4I \bar{\phi}) \partial_m \lambda, & \delta\mathcal{D}^{kl} &= 0. \end{aligned} \quad (4.7)$$

As is seen from (4.7), the gauge transformations of fields  $\phi, A_m, \Psi_\alpha^k$  are deformed due to the non-anticommutativity. In the limit  $I = 0$  we are left with the standard abelian gauge transformation for the vector potential  $A_m(x)$ ,  $\delta A_m(x) = \partial_m \lambda(x)$ .

The chiral coordinates are best suited for the chiral singlet deformation since the latter preserves chirality. In what follows we pass from the analytic coordinates  $\{x_A^m, \theta_\alpha^\pm, \bar{\theta}_\alpha^\pm\}$  to the mixed chiral-analytic ones  $\{z_C = (x_L^m, \theta_\alpha^\pm), \bar{\theta}_\alpha^\pm\}$  by the rule

$$x_A^m = x_L^m - 2i\theta^- \sigma^m \bar{\theta}^+. \quad (4.8)$$

For example, in the chiral-analytic basis the operator of chiral singlet deformations (2.24) is simplified to the form,

$$P_s = I(\overleftarrow{\partial}_+^\alpha \overrightarrow{\partial}_{-\alpha} - \overleftarrow{\partial}_-^\alpha \overrightarrow{\partial}_{+\alpha}), \quad (4.9)$$

where  $\partial_{\pm\alpha} = \partial/\partial\theta^{\pm\alpha}$ . Let us also rewrite the component structure of the prepotential (4.4) in these coordinates,

$$V_{WZ}^{++}(z_C, \bar{\theta}^+, u) = v^{++}(z_C, u) + \bar{\theta}_\alpha^+ v^{+\dot{\alpha}}(z_C, u) + (\bar{\theta}^+)^2 v(z_C, u). \quad (4.10)$$

Here

$$\begin{aligned} v^{++} &= (\theta^+)^2 \bar{\phi}, \\ v^{+\dot{\alpha}} &= -2\theta_\alpha^+ A^{\alpha\dot{\alpha}} + 4(\theta^+)^2 \bar{\Psi}^{-\dot{\alpha}} + 2i(\theta^+)^2 \theta_\alpha^- \partial^{\alpha\dot{\alpha}} \bar{\phi}, \\ v &= \phi + 4\theta^+ \Psi^- + 3(\theta^+)^2 \mathcal{D}^{--} - 2i(\theta^+ \theta^-) \partial_m A_m - \theta^- \sigma_{mn} \theta^+ F_{mn} \\ &\quad - (\theta^+)^2 (\theta^-)^2 \square \bar{\phi} + 4i(\theta^+)^2 \theta^- \sigma_m \partial_m \bar{\Psi}^- \end{aligned} \quad (4.11)$$

and  $F_{mn} = \partial_m A_n - \partial_n A_m$ .

Consider now the zero-curvature equation (3.2),

$$D^{++}V^{--} - D^{--}V_{WZ}^{++} + [V_{WZ}^{++}, V^{--}] = 0. \quad (4.12)$$

Developing the  $\star$ -product in (4.12) and applying (4.9), we have

$$\begin{aligned} &D^{++}V^{--} - D^{--}V_{WZ}^{++} + 2I(\partial_+^\alpha V_{WZ}^{++} \partial_{-\alpha} V^{--} - \partial_-^\alpha V_{WZ}^{++} \partial_{+\alpha} V^{--}) \\ &+ \frac{1}{2}I^3[\partial_-^\alpha (\partial_+)^2 V_{WZ}^{++} \partial_{+\alpha} (\partial_-)^2 V^{--} - \partial_+^\alpha (\partial_-)^2 V_{WZ}^{++} \partial_{-\alpha} (\partial_+)^2 V^{--}] = 0. \end{aligned} \quad (4.13)$$

We seek for a solution of the equation (4.13) as an expansion over  $\bar{\theta}_\alpha^\pm$ ,

$$\begin{aligned} V^{--} &= v^{--} + \bar{\theta}_\alpha^+ v^{-3\dot{\alpha}} + \bar{\theta}_\alpha^- v^{-\dot{\alpha}} + (\bar{\theta}^+)^2 v^{-4} + (\bar{\theta}^-)^2 \mathcal{A} + (\bar{\theta}^+ \bar{\theta}^-) \varphi^{--} \\ &\quad + \bar{\theta}^{+\dot{\alpha}} \bar{\theta}^{-\dot{\beta}} \varphi_{(\dot{\alpha}\dot{\beta})}^{-2} + (\bar{\theta}^-)^2 \bar{\theta}_\alpha^+ \tau^{-\dot{\alpha}} + (\bar{\theta}^+)^2 \bar{\theta}_\alpha^- \tau^{-3\dot{\alpha}} + (\bar{\theta}^+)^2 (\bar{\theta}^-)^2 \tau^{-2}, \end{aligned} \quad (4.14)$$

where  $v^{--}$ ,  $v^{-3\dot{\alpha}}$ ,  $v^{-4}$ ,  $\mathcal{A}$ ,  $\varphi^{--}$ ,  $\varphi_{(\dot{\alpha}\dot{\beta})}^{-2}$ ,  $\tau^{-\dot{\alpha}}$ ,  $\tau^{-3\dot{\alpha}}$ ,  $\tau^{-2}$  are the superfields depending only on the chiral-analytic variables  $x_L^m, \theta_\alpha^\pm, u_k^\pm$ . Note that the superfield  $\mathcal{A}$  that defines the classical SYM action (4.3) appears as one of the components in the expansion (4.14). Now we substitute the expressions (4.14), (4.10) into (4.13) and equate to zero the coefficients at the corresponding powers of  $\bar{\theta}_\alpha^\pm$ . In this way we obtain the following set of equations:

$$\mathbf{D}^{++}v^{--} - D^{--}v^{++} = 0, \quad (4.15)$$

$$\mathbf{D}^{++}v^{-\dot{\alpha}} - v^{+\dot{\alpha}} = 0, \quad (4.16)$$

$$\mathbf{D}^{++}v^{-3\dot{\alpha}} + v^{-\dot{\alpha}} - \mathbf{D}^{--}v^{+\dot{\alpha}} = 0, \quad (4.17)$$

$$\mathbf{D}^{++}\mathcal{A} = 0, \quad (4.18)$$

$$\mathbf{D}^{++}\varphi^{--} + 2\mathcal{A} - 2v + \frac{1}{2}\{v^{+\dot{\alpha}}; v_{\dot{\alpha}}^{-}\} = 0, \quad (4.19)$$

$$\mathbf{D}^{++}v^{-4} - \mathbf{D}^{--}v + \varphi^{--} + \frac{1}{2}\{v^{+\dot{\alpha}}; v_{\dot{\alpha}}^{-3}\} = 0, \quad (4.20)$$

$$\mathbf{D}^{++}\varphi_{(\dot{\alpha}\dot{\beta})}^{-2} + \frac{1}{2}\{v_{\dot{\alpha}}^{+}; v_{\dot{\beta}}^{-}\} + \frac{1}{2}\{v_{\dot{\beta}}^{+}; v_{\dot{\alpha}}^{-}\} = 0, \quad (4.21)$$

$$\mathbf{D}^{++}\tau^{-\dot{\alpha}} + [v^{+\dot{\alpha}}; \mathcal{A}] = 0, \quad (4.22)$$

$$\mathbf{D}^{++}\tau^{-3\dot{\alpha}} - \tau^{-\dot{\alpha}} + [v; v^{-\dot{\alpha}}] - \frac{1}{2}[v^{+\dot{\alpha}}; \varphi^{--}] + \frac{1}{2}[v_{\dot{\beta}}^{+}; \varphi^{-2(\dot{\alpha}\dot{\beta})}] = 0, \quad (4.23)$$

$$\mathbf{D}^{++}\tau^{-2} + \frac{1}{2}\{v^{+\dot{\alpha}}; \tau_{\dot{\alpha}}^{-}\} + [v; \mathcal{A}] = 0. \quad (4.24)$$

Here we used the notations

$$\mathbf{D}^{++} = D^{++} + [v^{++}; \cdot] = u_i^{+} \frac{\partial}{\partial u_i^{-}} + L\theta^{+\alpha}\partial_{-\alpha}, \quad (4.25)$$

$$\mathbf{D}^{--} = D^{--} + [v^{--}; \cdot] = u_i^{-} \frac{\partial}{\partial u_i^{+}} + \frac{1}{L}\theta^{-\alpha}\partial_{+\alpha}, \quad (4.26)$$

$$L = 1 + 4I\bar{\phi}. \quad (4.27)$$

It is straightforward (though somewhat lengthy) to find the solutions of (4.15)–(4.24),

$$v^{--}(z_C, u) = (\theta^{-})^2 \frac{\bar{\phi}}{L}, \quad (4.28)$$

$$v^{-3\dot{\alpha}}(z_C, u) = 2(\theta^{-})^2 \frac{\bar{\Psi}^{-\dot{\alpha}}}{L^2}, \quad (4.29)$$

$$v_{\dot{\alpha}}^{-}(z_C, u) = \frac{2}{L}\theta^{-\alpha}A_{\alpha\dot{\alpha}} - \frac{2}{L^2}(\theta^{-})^2\bar{\Psi}_{\dot{\alpha}}^{+} + \frac{4}{L}(\theta^{+}\theta^{-})\bar{\Psi}_{\dot{\alpha}}^{-} + \frac{2i}{L}(\theta^{-})^2\theta^{+\alpha}\partial_{\alpha\dot{\alpha}}\bar{\phi}, \quad (4.30)$$

$$\begin{aligned} \mathcal{A}(z_C, u) = & [\phi + \frac{4IA_mA_m}{L} + \frac{16I^3(\partial_m\bar{\phi})^2}{L}] \\ & + 2\theta^{+}[\Psi^{-} + \frac{4I(\sigma_m\bar{\Psi}^{-})A_m}{L}] - \frac{2\theta^{-}}{L}[\Psi^{+} + \frac{4I(\sigma_m\bar{\Psi}^{+})A_m}{L}] \\ & + (\theta^{+})^2[\frac{8I(\bar{\Psi}^{-})^2}{L} + \mathcal{D}^{--}] + \frac{(\theta^{-})^2}{L^2}[\frac{8I(\bar{\Psi}^{+})^2}{L} + \mathcal{D}^{++}] \\ & - \frac{2(\theta^{+}\theta^{-})}{L}[\frac{8I(\bar{\Psi}^{+}\bar{\Psi}^{-})}{L} + \mathcal{D}^{+-}] + (\theta^{+}\sigma_{mn}\theta^{-})(F_{mn} - \frac{8I\partial_{[m}\bar{\phi}A_{n]}}{L}) \\ & + 2i(\theta^{-})^2\theta^{+}\sigma_m\partial_m\frac{\bar{\Psi}^{+}}{L} + 2i(\theta^{+})^2L\theta^{-}\sigma_m\partial_m\frac{\bar{\Psi}^{-}}{L} - (\theta^{+})^2(\theta^{-})^2\Box\bar{\phi}, \end{aligned} \quad (4.31)$$

$$\begin{aligned} \varphi^{--}(z_C, u) = & \frac{4}{L}\theta^{-\alpha}\Psi_{\alpha}^{-} + \frac{8I}{L^2}\theta^{-\alpha}A_{\alpha\dot{\alpha}}\bar{\Psi}^{-\dot{\alpha}} + (\theta^{+}\theta^{-})[\frac{4}{L}\mathcal{D}^{--} + \frac{16I}{L^2}\bar{\Psi}_{\dot{\alpha}}^{-}\bar{\Psi}^{-\dot{\alpha}}] \\ & - (\theta^{-})^2[\frac{2i}{L}\partial_m A_m + \frac{2}{L^2}\mathcal{D}^{+-} + \frac{16I}{L^3}\bar{\Psi}_{\dot{\alpha}}^{-}\bar{\Psi}^{+\dot{\alpha}}] \\ & - 4i(\theta^{-})^2\theta^{+\alpha}\frac{1}{L}[\partial_{\alpha\dot{\alpha}}\bar{\Psi}^{-\dot{\alpha}} - \frac{2I}{L}\bar{\Psi}^{-\dot{\alpha}}\partial_{\alpha\dot{\alpha}}\bar{\phi}], \end{aligned} \quad (4.32)$$

$$v^{-4}(z_C, u) = (\theta^{-})^2[\frac{2}{L^2}\mathcal{D}^{--} + \frac{16I}{L^3}\bar{\Psi}_{\dot{\alpha}}^{-}\bar{\Psi}^{-\dot{\alpha}}], \quad (4.33)$$

$$\begin{aligned}
\varphi_{(\dot{\alpha}\dot{\beta})}^{-2}(z_C, u) &= -\frac{1}{2}\{v_{\dot{\alpha}}^{-} \star v_{\dot{\beta}}^{-}\} = 8IL^{-2}(\theta^{-\alpha}\bar{\Psi}_{\dot{\alpha}}^{-}A_{\alpha\dot{\beta}} + \theta^{-\alpha}\bar{\Psi}_{\dot{\beta}}^{-}A_{\alpha\dot{\alpha}}) \\
&\quad + 4iIL^{-2}(\theta^{-})^2(\partial_{\dot{\alpha}}^{\alpha}\bar{\phi}A_{\alpha\dot{\beta}} + \partial_{\dot{\beta}}^{\alpha}\bar{\phi}A_{\alpha\dot{\alpha}}) - 16IL^{-3}(\theta^{-})^2(\bar{\Psi}_{\dot{\beta}}^{+}\bar{\Psi}_{\dot{\alpha}}^{-} + \bar{\Psi}_{\dot{\alpha}}^{+}\bar{\Psi}_{\dot{\beta}}^{-}) \\
&\quad + 8iIL^{-2}(\theta^{-})^2\theta^{+\alpha}(\bar{\Psi}_{\dot{\alpha}}^{-}\partial_{\alpha\dot{\beta}}\bar{\psi} + \bar{\Psi}_{\dot{\beta}}^{-}\partial_{\alpha\dot{\alpha}}\bar{\phi}), \tag{4.34}
\end{aligned}$$

$$\tau^{-\dot{\alpha}} = [\mathcal{A} \star v^{-\dot{\alpha}}], \tag{4.35}$$

$$\tau^{-3\dot{\alpha}} = -\mathbf{D}^{--}\tau^{-\dot{\alpha}} + \frac{1}{2}[v^{-\dot{\alpha}} \star \varphi^{--}] - \frac{1}{2}[v_{\dot{\beta}}^{-} \star \varphi^{-2(\dot{\alpha}\dot{\beta})}], \tag{4.36}$$

$$\tau^{-2} = -\frac{1}{2}[\varphi^{--} \star \mathcal{A}] - \frac{1}{4}\{v^{-\dot{\alpha}} \star [\mathcal{A} \star v_{\dot{\alpha}}^{-}]\}. \tag{4.37}$$

The expressions (4.35)–(4.37) are presented in a superfield form since their exact component structure is of no importance for our further consideration.

Now we use the expression (4.31) to find the component structure of the classical action (4.3):

$$S_{SYM} = S_{\phi} + S_{\Psi} + S_A, \tag{4.38}$$

$$S_{\phi} = -\frac{1}{2} \int d^4x \square \bar{\phi} \left[ \phi + \frac{4IA_m A_m}{1+4I\bar{\phi}} + \frac{16I^3 \partial_m \bar{\phi} \partial_m \bar{\phi}}{1+4I\bar{\phi}} \right], \tag{4.39}$$

$$\begin{aligned}
S_{\Psi} &= i \int d^4x \left( \Psi^{i\alpha} + \frac{4IA_m \sigma_m^{\alpha}{}_{\dot{\alpha}} \bar{\Psi}^{i\dot{\alpha}}}{1+4I\bar{\phi}} \right) (\sigma_n)_{\alpha\dot{\beta}} \partial_n \left( \frac{\bar{\Psi}_i^{\dot{\beta}}}{1+4I\bar{\phi}} \right) \\
&\quad + \frac{1}{4} \int d^4x \frac{1}{(1+4I\bar{\phi})^2} \left( \frac{8I\bar{\Psi}_{\dot{\alpha}}^i \bar{\Psi}^{j\dot{\alpha}}}{1+4I\bar{\phi}} + \mathcal{D}^{ij} \right) \left( \frac{8I\bar{\Psi}_{i\dot{\alpha}} \bar{\Psi}_j^{\dot{\alpha}}}{1+4I\bar{\phi}} + \mathcal{D}_{ij} \right), \tag{4.40}
\end{aligned}$$

$$\begin{aligned}
S_A &= \int d^4x \left[ -\frac{1}{2} A_n \square A_n - \frac{1}{2} \partial_m A_m \partial_n A_n + \frac{1}{2} A_n A_n \square \ln(1+4I\bar{\phi}) \right. \\
&\quad \left. - \varepsilon_{mnrs} \partial_r A_s A_n \partial_m \ln(1+4I\bar{\phi}) + \frac{1}{2} A_n A_n \partial_m \ln(1+4I\bar{\phi}) \partial_m \ln(1+4I\bar{\phi}) \right. \\
&\quad \left. - \frac{1}{2} A_m A_n \partial_m \ln(1+4I\bar{\phi}) \partial_n \ln(1+4I\bar{\phi}) + \partial_n A_m A_n \partial_m \ln(1+4I\bar{\phi}) \right]. \tag{4.41}
\end{aligned}$$

Let us make two comments on the symmetries of the action (4.38). First of all, it is invariant under the gauge transformations (4.7). Secondly, it respects the residual  $\mathcal{N}=(1,0)$  supersymmetry,

$$\begin{aligned}
\delta_{\epsilon} \phi &= 2(\epsilon^k \Psi_k), & \delta_{\epsilon} \bar{\phi} &= 0, \\
\delta_{\epsilon} A_m &= (\epsilon^k \sigma_m \bar{\Psi}_k), \\
\delta_{\epsilon} \Psi_{\alpha}^k &= -\epsilon_{\alpha l} \mathcal{D}^{kl} + \frac{1}{2}(1+4I\bar{\phi})(\sigma_{mn} \epsilon^k)_{\alpha} F_{mn} - 4iI\epsilon_{\alpha}^k A_m \partial_m \bar{\phi}, \\
\delta_{\epsilon} \bar{\Psi} &= -i(1+4I\bar{\phi})(\epsilon^k \sigma_m)_{\dot{\alpha}} \partial_m \bar{\phi}, \\
\delta_{\epsilon} \mathcal{D}^{kl} &= i\partial_m [(\epsilon^k \sigma_m \bar{\Psi}^l + \epsilon^k \sigma_m \bar{\Psi}^l)(1+4I\bar{\phi})]. \tag{4.42}
\end{aligned}$$

We observe that both gauge transformations (4.7) and the supersymmetry (4.42) are deformed by the non-anticommutativity parameter  $I$ .

It is well known that the classical action of the undeformed supergauge theory can be equivalently written in either chiral or antichiral superspace, because of the equality <sup>5</sup>

$$\frac{1}{4}\text{tr} \int d^4x_L d^4\theta W^2 = \frac{1}{4}\text{tr} \int d^4x_R d^4\bar{\theta} \bar{W}^2. \quad (4.43)$$

Surprisingly, the relation (4.43) fails to be valid in the  $\mathcal{N}=(1,0)$  non-anticommutative supergauge model. Moreover, it is inconsistent to treat the expression  $\frac{1}{4}\text{tr} \int d^4x_R d^4\bar{\theta} \bar{W}^2$  as any action since it bears the explicit dependence on Grassmann and harmonic variables.

This statement can be most easily proved in the abelian case. To this end, we consider the covariantly antichiral superfield strength in the antichiral coordinates (A.21),

$$\bar{W} = -\frac{1}{4}(D^+)^2 V^{--} = \bar{\mathcal{A}} + \theta^{+\alpha} \bar{\tau}_\alpha^- + (\theta^+)^2 \bar{\tau}^{-2}, \quad (4.44)$$

where  $\bar{\mathcal{A}}$ ,  $\bar{\tau}_\alpha^-$  and  $\bar{\tau}^{-2}$  are purely antichiral superfields defined on the coordinate set  $x_R^m, \bar{\theta}_\alpha^\pm, u^\pm$ . This superfield strength, as well as the prepotential  $V^{--}$ , depend on the parameter of non-anticommutativity  $I$ . Let us expand  $\bar{W}$  in powers of  $I$

$$\bar{W} = \sum_{n=0}^{\infty} I^n \bar{W}_n, \quad (4.45)$$

where the coefficients  $\bar{W}_n$  are some superfields. Clearly, the first term  $\bar{W}_0$  in this series is a purely antichiral superfield which has the same component structure as the undeformed superfield strength,

$$\begin{aligned} \bar{W}_0 = \bar{\mathcal{A}}_0 = & \bar{\phi} - 2\bar{\theta}_\alpha^+ \bar{\Psi}^{-\dot{\alpha}} + 2\bar{\theta}_\alpha^- \bar{\Psi}^{+\dot{\alpha}} + i\bar{\theta}^{-\dot{\alpha}} \bar{\theta}^{+\dot{\beta}} (\partial_{\dot{\alpha}}^\alpha A_{\alpha\dot{\beta}} + \partial_{\dot{\alpha}}^\alpha A_{\alpha\dot{\beta}}) \\ & + (\bar{\theta}^+)^2 \mathcal{D}^{--} + (\bar{\theta}^-)^2 \mathcal{D}^{++} - 2(\bar{\theta}^+ \bar{\theta}^-) \mathcal{D}^{+-} \\ & - 2i(\bar{\theta}^-)^2 \bar{\theta}^{+\dot{\alpha}} u_k^+ \partial_{\alpha\dot{\alpha}} \Psi^{\alpha k} - 2i(\bar{\theta}^+)^2 \bar{\theta}^{-\dot{\alpha}} u_k^- \partial_{\alpha\dot{\alpha}} \Psi^{\alpha k} - (\bar{\theta}^+)^2 (\bar{\theta}^-)^2 \square \phi. \end{aligned} \quad (4.46)$$

Note that  $\bar{W}_0$  is harmonic-independent,  $D^{\pm\pm} \bar{W}_0 = 0$ , whereas the next term  $\bar{W}_1$  bears such a dependence,

$$I D^{++} \bar{W}_1 = -[V^{++} \star \bar{W}_0] \neq 0. \quad (4.47)$$

Now we are going to prove that the expression

$$A = \int d^4x_R d^4\bar{\theta} \bar{W}_\star^2 \quad (4.48)$$

depends on Grassmann variables and harmonics,

$$D^{\pm\pm} A \neq 0, \quad D_\alpha^- A \neq 0. \quad (4.49)$$

For this purpose we expand it in powers of  $I$ ,

$$A = \sum_{n=0}^{\infty} I^n A_n, \quad (4.50)$$

---

<sup>5</sup>Note that the analogous relation for the  $\mathcal{N}=1$  supersymmetric theories reads  $\int d^4x d^2\theta W^\alpha W_\alpha = \int d^4x d^2\bar{\theta} \bar{W}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}}$ , where  $W_\alpha$ ,  $\bar{W}_{\dot{\alpha}}$  are the  $\mathcal{N}=1$  gauge superfield strengths. As shown in [12], this relation also holds in the corresponding non-anticommutative gauge theory with  $\mathcal{N}=(1/2,0)$  supersymmetry.

and check the inequalities (4.49) in the first order in  $I$ . Up to terms of the second order in  $I$  we have

$$A_0 + IA_1 = \int d^4x_R d^4\bar{\theta} (\bar{W}_0^2 + 2I\bar{W}_0\bar{W}_1). \quad (4.51)$$

Clearly, the term  $A_0 = \int d^4x_R d^4\bar{\theta} \bar{W}_0^2$  in (4.51) does not depend on harmonics since  $D^{\pm\pm}W_0 = 0$ . Therefore we have to consider only the harmonic derivative of  $A_1$  which is given by

$$D^{++}A_1 = 4i \int d^4x_R d^4\bar{\theta} \partial_+^\alpha V^{++} \bar{\theta}^{+\dot{\alpha}} \partial_{\alpha\dot{\alpha}} (\bar{W}_0^2). \quad (4.52)$$

It is a technical exercise to derive the component structure of  $A_1$ , given the component expansions (4.4), (4.46) of the superfields  $V^{++}$  and  $\bar{W}_0$ . It is sufficient to consider only two terms in the expression  $\partial_+^\alpha V^{++} \bar{\theta}^{+\dot{\alpha}} = (\bar{\theta}^+)^2 A^{\alpha\dot{\alpha}} + 2\theta^{+\alpha} \bar{\theta}^{+\dot{\alpha}} \bar{\phi} + \dots$  to come to the conclusion that

$$D^{++}A_1 = 16i \int d^4x_R [A_m \partial_m (\bar{\phi} D^{++}) + \theta^{+\alpha} \bar{\phi} \partial_{\alpha\dot{\beta}} (\bar{\phi} \partial^{\beta\dot{\beta}} \Psi_\beta^+)] + \dots \neq 0. \quad (4.53)$$

The terms written down in (4.53) cannot be cancelled by any other ones (which are omitted here). The manifest dependence on harmonics implied by (4.53) entails also the dependence on  $\theta^+$  variables owing to the commutation relation  $[D^{--}, D_\alpha^+] = D_\alpha^-$ . Therefore, (4.53) proves the inequalities (4.49) which show that  $\text{tr} \int d^4x_R d^4\bar{\theta} \bar{W}^2$  cannot be treated as a superfield action.

It is easy to argue that the more general expression  $\int d^4x_R d^4\bar{\theta} \bar{\mathcal{F}}_\star(\bar{W})$  also involves a manifest dependence on the Grassmann variables and harmonics. This implies that among the candidate contributions to the effective action of the supersymmetric gauge model there are no such ones which are given by integrals of some functions of the superfield strength  $\bar{W}$  over the antichiral superspace. In Sect. 6 we will demonstrate that the contributions to the effective action in the non-anticommutative case are naturally written as integrals over the full  $\mathcal{N}=(1,1)$  superspace.

## 4.2 Seiberg-Witten transform in the abelian supergauge model

The equations (4.7), (4.42) show that both gauge and supersymmetry transformations depend on the parameter of the chiral singlet deformation  $I$ . A natural question is whether there exist any change of the variables in the functional integral which would bring these transformations to the undeformed form. For example, for the gauge models with the bosonic noncommutative deformation such a transformation was found in [9], and it is known as the Seiberg-Witten map. Remarkably, for the chiral singlet deformation such a field redefinition also exists. It was found in [26, 31]:

$$\begin{aligned} \phi &\rightarrow \varphi = \frac{1}{(1+4I\bar{\phi})^2} \left[ \phi + \frac{4I(A_m A_m + 4I^2 \partial_m \bar{\phi} \partial_m \bar{\phi})}{1+4I\bar{\phi}} \right], \\ A_m &\rightarrow a_m = \frac{A_m}{1+4I\bar{\phi}}, \quad \bar{\Psi}_\alpha^k \rightarrow \bar{\psi}_\alpha^k = \frac{\bar{\Psi}_\alpha^k}{1+4I\bar{\phi}}, \\ \Psi_\alpha^k &\rightarrow \psi_\alpha^k = \frac{1}{(1+4I\bar{\phi})^2} \left[ \Psi_\alpha^k + \frac{4I A_{\alpha\dot{\alpha}} \bar{\Psi}^{\dot{\alpha}k}}{1+4I\bar{\phi}} \right], \\ \mathcal{D}^{kl} &\rightarrow d^{kl} = \frac{1}{(1+4I\bar{\phi})^2} \left[ \mathcal{D}^{kl} + \frac{8I \bar{\Psi}_\alpha^k \bar{\Psi}^{\dot{\alpha}l}}{1+4I\bar{\phi}} \right]. \end{aligned} \quad (4.54)$$

It is easy to check that the supertranslations (4.42), being rewritten in terms of the fields (4.54), read

$$\begin{aligned}
\delta_\epsilon \varphi &= 2(\epsilon^k \psi_k), & \delta_\epsilon \bar{\phi} &= 0, \\
\delta_\epsilon a_m &= (\epsilon^k \sigma_m \bar{\psi}_k), \\
\delta_\epsilon \psi_\alpha^k &= -\epsilon_{\alpha l} d^{kl} + \frac{1}{2}(\sigma_{mn} \epsilon^k)_\alpha f_{mn}, \\
\delta_\epsilon \bar{\psi}_{\dot{\alpha}}^k &= -i(\epsilon^k \sigma_m)_{\dot{\alpha}} \partial_m \bar{\phi}, \\
\delta_\epsilon d^{kl} &= i\partial_m (\epsilon^k \sigma_m \bar{\psi}^l + \epsilon^k \sigma_m \bar{\psi}^l),
\end{aligned} \tag{4.55}$$

where  $f_{mn} = \partial_m a_n - \partial_n a_m$ . The gauge transformations of the fields (4.54) also coincide with those for the undeformed fields. Namely, all fields are the gauge group singlets, except for  $a_m$  which transforms as

$$\delta_r a_m = \partial_m \lambda. \tag{4.56}$$

Surprisingly, the field redefinition (4.54) drastically simplifies the structure of the action (4.38). In terms of fields  $\varphi$ ,  $\bar{\phi}$ ,  $\psi_k^\alpha$ ,  $\bar{\psi}^{\dot{\alpha}k}$ ,  $a_m$ ,  $d^{kl}$  it is given by

$$S_{SYM} = \int d^4x \mathcal{L} = \int d^4x (1 + 4I\bar{\phi})^2 \mathcal{L}_0, \tag{4.57}$$

where

$$\mathcal{L}_0 = -\frac{1}{2}\varphi \square \bar{\phi} + \frac{1}{4}(f_{mn}f_{mn} + \frac{1}{2}\varepsilon_{mnrs}f_{mn}f_{rs}) - i\psi_k^\alpha \partial_{\alpha\dot{\alpha}} \bar{\psi}^{\dot{\alpha}k} + \frac{1}{4}d^{kl}d_{kl}. \tag{4.58}$$

The expression  $\mathcal{L}_0$  is none other than a Lagrangian of  $\mathcal{N}=(1,1)$  supersymmetric U(1) gauge theory. As a result, the net effect of the chiral singlet deformation in terms of the new fields is the appearance of the factor  $(1 + 4I\bar{\phi})^2$  in front of the undeformed Lagrangian.

It is also worth pointing out that the Seiberg-Witten map is not unique. Indeed, since the scalar field  $\bar{\phi}$  is a singlet of both the gauge transformations and  $\mathcal{N}=(1,0)$  supersymmetry, one can rescale the fields as

$$\hat{\varphi} = L^2 \varphi, \quad \hat{\psi}_\alpha^k = L^2 \psi_\alpha^k, \quad \hat{d}^{kl} = L d^{kl}, \tag{4.59}$$

which does not affect gauge transformations and supersymmetry. When written in terms of the fields (4.59), the Lagrangian  $\mathcal{L}$  takes the most simple form,

$$\mathcal{L} = -\frac{1}{2}\hat{\varphi} \square \bar{\phi} + \frac{1}{4}L^2(f_{mn}f_{mn} + \frac{1}{2}\varepsilon_{mnrs}f_{mn}f_{rs}) - i\hat{\psi}_k^\alpha \partial_{\alpha\dot{\alpha}} \bar{\psi}^{\dot{\alpha}k} + \frac{1}{4}\hat{d}^{kl}\hat{d}_{kl}. \tag{4.60}$$

We see that the only remaining interaction is that between the gauge field strength  $f_{mn}$  and the scalar field  $\bar{\phi}$ . The lagrangian (4.60) is bilinear in all other fields, like in the free case.

Let us now discuss the problem of a superfield representation for the Seiberg-Witten-like map (4.54). For this purpose we need a relation between the superfield  $\mathcal{A}$  given by (4.31) and the undeformed superfield strength  $W_0$  given by the expression

$$\begin{aligned}
W_0(x_L, \theta^+, \theta^-, u) &= \varphi + 2\theta^+ \psi^- - 2\theta^- \psi^+ + (\theta^- \sigma_{mn} \theta^+) f_{mn} \\
&\quad + (\theta^+)^2 d^{--} + (\theta^-)^2 d^{++} - 2(\theta^+ \theta^-) d^{+-} \\
&\quad + 2i(\theta^-)^2 \theta^+ \sigma_m \partial_m \bar{\psi}^+ + 2i(\theta^+)^2 \theta^- \sigma_m \partial_m \bar{\psi}^- - (\theta^+)^2 (\theta^-)^2 \square \bar{\phi}.
\end{aligned} \tag{4.61}$$



Here we use the notation  $\psi_\alpha^\pm = \psi_\alpha^i u_i^\pm$ ,  $d^{+-} = u_k^+ u_l^- d^{kl}$ , etc, as for the original fields. By definition, the superfield strength (4.61) is gauge invariant,

$$\delta_\lambda W_0 = 0, \quad (4.62)$$

and it transforms under  $\mathcal{N}=(1,0)$  supersymmetry in the standard way,

$$\delta_\epsilon W_0 = (\epsilon^{-\alpha} \partial_{-\alpha} + \epsilon^{+\alpha} \partial_{+\alpha}) W_0. \quad (4.63)$$

There is a simple relation between the expressions (4.31) and (4.61) [26],

$$\mathcal{A}(x_L, \theta^+, \theta^-, u) = (1 + 4I\bar{\phi})^2 W_0(x_L, \theta^+, (1 + 4I\bar{\phi})^{-1} \theta^-, u). \quad (4.64)$$

The equation (4.64) plays the role of the superfield Seiberg-Witten transform. It is essential that, up to an overall scalar factor, it amounts to rescaling the variable  $\theta^-$  by the factor  $(1 + 4I\bar{\phi})^{-1}$ .

Let us now introduce the following differential operator

$$R_\theta = \exp(L^{-1} \theta^- \partial_-) = L^{-2} + \partial_{-\alpha} \{1 - L^{-1} - \frac{1}{4}(L^{-1} - 1)^2 [2\theta^{-\alpha} - (\theta^-)^2 \partial_-^\alpha]\}, \quad (4.65)$$

where  $L = 1 + 4I\bar{\phi}$ . Using this operator, the superfield Seiberg-Witten transform (4.64) can be rewritten as

$$\mathcal{A} = L^2 R_\theta W_0. \quad (4.66)$$

Owing to the simple property  $R_\theta A R_\theta B = R_\theta (AB)$ , we have

$$\mathcal{A}^2 = L^4 R_\theta W_0^2. \quad (4.67)$$

Employing now the relations (4.65), (4.67), one easily constructs the Seiberg-Witten transform of the classical action (4.3),

$$S_{SYM} = \frac{1}{4} \int d^4 x_L d^4 \theta \mathcal{A}^2 = \frac{1}{4} \int d^4 x_L d^4 \theta (1 + 4I\bar{\phi})^2 W_0^2. \quad (4.68)$$

This is just the action (4.57) derived before.

The Seiberg-Witten transform found here for the classical action (4.68) can be readily generalized to the action with an arbitrary chiral potential,

$$\int d^4 x_L d^4 \theta \mathcal{F}_\star(\mathcal{A}), \quad (4.69)$$

where  $\mathcal{F}_\star(\mathcal{A})$  is some function given by a series,

$$\mathcal{F}_\star(\mathcal{A}) = \sum_{n=2}^{\infty} c_n \mathcal{A}_\star^n. \quad (4.70)$$

The function  $\mathcal{A}_\star^n$  is expressed through the undeformed superfield strength (4.61) as follows

$$\mathcal{A}_\star^n = L^{2n-2} (W_0)_\star^n + \dots, \quad (4.71)$$

where we introduced a modified  $\star$ -product,

$$\hat{\star} = \exp(L^{-1}P_s). \quad (4.72)$$

Dots in (4.71) stand for terms which involve full spinor derivatives  $\partial_{-\alpha}$  coming from the expansion of the operator (4.65). These terms are not essential when they are considered under the integral over chiral superspace. As a result, the action (4.69) is expressed through the undeformed superfield strengths (4.61),

$$\int d^4x_L d^4\theta \mathcal{F}_\star(\mathcal{A}) = \int d^4x_L d^4\theta L^{-2} \mathcal{F}_\star(L^2 W_0). \quad (4.73)$$

Here, the function  $\mathcal{F}_\star(L^2 W_0)$  is given by the series (4.70) with the  $\hat{\star}$ -product of superfields. The relation (4.73) plays the role of Seiberg-Witten transform for the chiral effective action.

Let us point out that the choice (4.59) not only brings the classical action to the most simple form but also is very useful for studying the contributions to chiral effective potentials. In particular, given the expansion of the superfield  $\mathcal{A}$  in terms of these fields,

$$\begin{aligned} \mathcal{A} = & \hat{\varphi} + 2\theta^{+\alpha} \hat{\psi}_\alpha^- - 2L^{-1}\theta^{-\alpha} \hat{\psi}_\alpha^+ + L(\theta^+ \sigma_{mn} \theta^-) f_{mn} \\ & + L(\theta^+)^2 \hat{d}^{--} - 2(\theta^+ \theta^-) \hat{d}^{+-} + L^{-1}(\theta^-)^2 \hat{d}^{++} \\ & + 2i[(\theta^-)^2 \theta^{+\alpha} \partial_{\alpha\dot{\alpha}} \bar{\psi}^{+\dot{\alpha}} + L(\theta^+)^2 \theta^{-\alpha} \partial_{\alpha\dot{\alpha}} \bar{\psi}^{-\dot{\alpha}}] - (\theta^+)^2 (\theta^-)^2 \square \bar{\phi}, \end{aligned} \quad (4.74)$$

one can readily find the component structure of cubic and quartic terms in the effective potential  $\mathcal{F}_\star(\mathcal{A})$  in the bosonic sector,

$$\begin{aligned} \int d^4\theta \mathcal{A}_\star^3 = & -3\hat{\varphi}^2 \square \bar{\phi} + 3\hat{\varphi}(\hat{d}_{kl})^2 + \frac{3}{4}L^2 \hat{\varphi}(f_{\alpha\beta})^2 \\ & - 3I^2 \square \bar{\phi} [L^2(f_{\alpha\beta})^2 - 4(\hat{d}_{kl})^2] - 16I^4(\square \bar{\phi})^3, \end{aligned} \quad (4.75)$$

$$\begin{aligned} \int d^4\theta \mathcal{A}_\star^4 = & -4\hat{\varphi}^3 \square \bar{\phi} + 6\hat{\varphi}^2(\hat{d}_{kl})^2 + \frac{3}{2}L^2 \hat{\varphi}^2(f_{\alpha\beta})^2 \\ & + 2I^2[L^2(f_{\alpha\beta})^2 - 4(\hat{d}_{kl})^2][ -6\hat{\varphi} \square \bar{\phi} + (\hat{d}_{kl})^2 + \frac{1}{4}L^2(f_{\alpha\beta})^2] \\ & + 8I^4(\square \bar{\phi})^2[3L^2(f_{\alpha\beta})^2 + 12(\hat{d}_{kl})^2 - 8\hat{\varphi} \square \bar{\phi}], \end{aligned} \quad (4.76)$$

where  $(f_{\alpha\beta})^2 = f^{\alpha\beta} f_{\alpha\beta} = (f_{mn})^2 + f_{mn} \tilde{f}_{mn}$  and  $(\hat{d}_{kl})^2 = \hat{d}^{kl} \hat{d}_{kl}$ . These expressions are the chiral singlet deformation of the corresponding terms  $\int d^4\theta W_0^3$  and  $\int d^4\theta W_0^4$  in the undeformed  $\mathcal{N}=(1,1)$  holomorphic effective action. The equations (4.75), (4.76) show that the chiral singlet deformation manifests itself not only in the appearance of induced interaction of vector field with scalars but also in the presence of new terms with the field derivatives. Another important consequence of the non-anticommutative deformation of the effective potentials is the appearance of non-linear self-coupling of the auxiliary fields  $I^2(\hat{d}_{kl})^4$ .

### 4.3 Neutral hypermultiplet model

The  $\mathcal{N}=(1,1)$  hypermultiplet on-shell content is four real scalar fields  $f^{ak}$ ,  $a, k = 1, 2$  and two independent spinors  $\rho^{aa}, \chi_a^{\dot{a}}$ . Here we assume that these fields are neutral with respect to the

U(1) group. We are going to find the interactions of these fields with the vector multiplet  $\phi$ ,  $\bar{\phi}$ ,  $\Psi_k^\alpha$ ,  $\bar{\Psi}^{\dot{\alpha}k}$ ,  $D^{kl}$ .

The classical action of the neutral hypermultiplet is given by (3.21). Taking into account the equations (3.19) and (2.37), we make explicit the  $\star$ -product in (3.21),

$$S_{ad}[q^+, V^{++}] = \frac{1}{2} \int d\zeta du q_a^+ [D^{++} q^{+a} + 4iI\bar{\theta}^{+\dot{\alpha}}(\partial_+^\alpha V^{++} \partial_{\alpha\dot{\alpha}} q^{+a} - \partial_{\alpha\dot{\alpha}} V^{++} \partial_+^\alpha q^{+a})]. \quad (4.77)$$

The hypermultiplet superfield has the following component field expansion

$$\begin{aligned} q^{+a} = & f^{+a} + \theta^{+\alpha} \pi_\alpha^a + \bar{\theta}_\alpha^{+\dot{\alpha}} \kappa^{\dot{\alpha}a} + \theta^+ \sigma_m \bar{\theta}^+ r_m^{-a} + (\theta^+)^2 g^{-a} + (\bar{\theta}^+)^2 h^{-a} \\ & + (\bar{\theta}^+)^2 \theta^{+\alpha} \Sigma_\alpha^{-a} + (\theta^+)^2 \bar{\theta}^{+\dot{\alpha}} \bar{\Sigma}_{\dot{\alpha}}^{-a} + (\theta^+)^2 (\bar{\theta}^+)^2 \omega^{-3a}, \end{aligned} \quad (4.78)$$

where all the component fields depend only on the variables  $x_A^m$  and  $u_i^\pm$ . These fields can be further expanded over the harmonic variables, giving rise to an infinite number of the auxiliary fields. The auxiliary fields should be eliminated from the action using the classical equation of motion for the hypermultiplet superfield  $q^{+a}$ . This equation is easily obtained from the action (4.77),

$$D^{++} q^{+a} + 4iI\bar{\theta}^{+\dot{\alpha}}(\partial_+^\alpha V^{++} \partial_{\alpha\dot{\alpha}} q^{+a} - \partial_{\alpha\dot{\alpha}} V^{++} \partial_+^\alpha q^{+a}) = 0. \quad (4.79)$$

Substituting (4.78), (4.4) into (4.79) we find the explicit expressions of the hypermultiplet component fields in terms of the physical scalars  $f^{ia}$  and fermions  $\rho_\alpha^a$ ,  $\chi^{\dot{\alpha}a}$ ,

$$\begin{aligned} f^{+a} &= f^{ak} u_k^+, \quad \pi_\alpha^a = \rho_\alpha^a, \quad \kappa^{\dot{\alpha}a} = \chi^{\dot{\alpha}a}, \quad r_m^{-a} = r_m^{ak} u_k^-, \quad g^{-a} = 0, \\ h^{-a} &= h^{ak} u_k^-, \quad \Sigma_\alpha^{-a} = \Sigma_\alpha^{kla} u_k^- u_l^-, \quad \bar{\Sigma}_{\dot{\alpha}}^{-a} = 0, \quad \omega^{-3a} = 0, \\ r_m^{ak} &= 2i(1 + 4I\bar{\phi}) \partial_m f^{ak}, \quad h^{ak} = -8iI A_m \partial_m f^{ak}, \\ \Sigma_\alpha^{kla} &= -4iI(\bar{\Psi}^{\dot{\alpha}k} \partial_{\alpha\dot{\alpha}} f^{al} + \bar{\Psi}^{\dot{\alpha}l} \partial_{\alpha\dot{\alpha}} f^{ak}). \end{aligned} \quad (4.80)$$

Taking into account (4.80), we obtain the following action for the physical components in the neutral hypermultiplet model,

$$\begin{aligned} S_{ad} = & \int d^4x \left[ \frac{1}{2} (1 + 4I\bar{\phi})^2 \partial_m f^{ak} \partial_m f_{ak} + \frac{1}{2} i (1 + 4I\bar{\phi}) \rho^{\alpha a} \partial_{\alpha\dot{\alpha}} \chi_a^{\dot{\alpha}} \right. \\ & \left. + 4iI \bar{\Psi}_k^{\dot{\alpha}} \rho_a^\alpha \partial_{\alpha\dot{\alpha}} f^{ak} + 2iI \rho^{\alpha a} A_m \partial_m \rho_{\alpha a} + iI \rho^{\beta a} \rho_a^\alpha \partial_{(\alpha\dot{\alpha}} A_{\dot{\beta})}^{\dot{\alpha}} \right]. \end{aligned} \quad (4.81)$$

Note that only the fields  $\bar{\phi}$ ,  $A_m$ ,  $\bar{\Psi}_k^{\dot{\alpha}}$  from the vector multiplet interact with the hypermultiplet fields in (4.81).

Let us study the symmetries of the action (4.81). This action is invariant in the evident way under the gauge transformations (3.16). Using the relation (2.37), these gauge transformations can be cast in the following form

$$\delta_\Lambda q^{+a} = 4i\bar{\theta}_\alpha^{+\dot{\alpha}} (\partial^{\alpha\dot{\alpha}} \Lambda \partial_{+\alpha} q^{+a} - \partial_{+\alpha} \Lambda \partial^{\alpha\dot{\alpha}} q^{+a}). \quad (4.82)$$

Recall that the component structure of the vector multiplet (4.4) is given in the Wess-Zumino gauge. Therefore it makes sense to discuss here only the residual gauge transformations with the parameter  $\Lambda_r = i\lambda(x_A)$ . With such a choice of the gauge parameter the hypermultiplet gauge transformations (4.82) are reduced to

$$\delta_r q^{+a} = -4I\bar{\theta}_\alpha^{+\dot{\alpha}} \partial^{\alpha\dot{\alpha}} \lambda(x_A) \partial_{+\alpha} q^{+a}. \quad (4.83)$$

Equation (4.83) leads to the following gauge transformations of the hypermultiplet component fields

$$\delta_r f^{ak} = 0, \quad \delta_r \rho_\alpha^a = 0, \quad \delta_r \chi^{\dot{\alpha}a} = -4I \partial^{\alpha\dot{\alpha}} \lambda \rho_\alpha^a. \quad (4.84)$$

Let us also consider the  $\mathcal{N}=(1,0)$  supersymmetry transformations for the hypermultiplet

$$\delta_\epsilon q^{+a} = (\epsilon^{-\alpha} Q_\alpha^+ - \epsilon^{+\alpha} Q_\alpha^-) q^{+a} = (\epsilon^{+\alpha} \partial_{+\alpha} - 2i\epsilon^{-\alpha} \bar{\theta}^{+\dot{\alpha}} \partial_{\alpha\dot{\alpha}}) q^{+a}. \quad (4.85)$$

In the Wess-Zumino gauge (4.85) leads to the following component field transformations

$$\delta_\epsilon f^{ak} = \epsilon^{ak} \rho_\alpha^a, \quad \delta_\epsilon \rho_\alpha^a = 0, \quad \delta_\epsilon \chi_{\dot{\alpha}}^a = 2i\epsilon^{\alpha k} (1 + 4I\bar{\phi}) \partial_{\alpha\dot{\alpha}} f_k^a. \quad (4.86)$$

It is an easy exercise to check that the action (4.81) is invariant with respect to the hypermultiplet gauge transformations (4.84) combined with (4.7), as well as with respect to the  $\mathcal{N}=(1,0)$  supersymmetry (4.86) combined with (4.42).

Both supersymmetry transformations (4.85) and the gauge transformations (4.84) are deformed due to the explicit presence of the parameter  $I$ . Let us consider the following transform of the hypermultiplet fields

$$\begin{aligned} f^{ak} &\rightarrow f_0^{ak} = (1 + 4I\bar{\phi}) f^{ak}, \\ \rho^{\alpha a} &\rightarrow \rho_0^{\alpha a} = (1 + 4I\bar{\phi}) \rho^{\alpha a}, \\ \chi^{\dot{\alpha}a} &\rightarrow \chi_0^{\dot{\alpha}a} = \chi^{\dot{\alpha}a} + \frac{4I A^{\alpha\dot{\alpha}} \rho_\alpha^a}{1 + 4I\bar{\phi}} - \frac{8I \bar{\Psi}^{\dot{\alpha}k} f_k^a}{1 + 4I\bar{\phi}}. \end{aligned} \quad (4.87)$$

One can easily check that the new fields  $f_0^{ak}$ ,  $\rho_0^{\alpha a}$ ,  $\chi_0^{\dot{\alpha}a}$  transform under  $U(1)$  gauge group and  $\mathcal{N}=(1,0)$  supersymmetry in the standard way, i.e. as in the undeformed case with  $I = 0$ . Therefore we can refer to the transformation (4.87) as a Seiberg-Witten map for the neutral hypermultiplet model. In terms of the fields  $f_0^{ak}$ ,  $\rho_0^{\alpha a}$ ,  $\chi_0^{\dot{\alpha}a}$  the action (4.81) is rewritten as

$$\begin{aligned} S_{ad} = & \int d^4x \left[ \frac{1}{2} \partial_m f_0^{ak} \partial_m f_{0ak} + \frac{i}{2} \rho_0^{\alpha a} \partial_{\alpha\dot{\alpha}} \chi_0^{\dot{\alpha}a} + \frac{2iI \rho_0^{\beta a} \rho_{0a}^\alpha \partial_{(\alpha\dot{\alpha}} a_{\beta)}^{\dot{\alpha}}}{1 + 4I\bar{\phi}} \right. \\ & \left. + \frac{2I f_0^{ak} f_{0ak} \square \bar{\phi}}{1 + 4I\bar{\phi}} + \frac{4iI \rho_0^{\alpha a} f_{0ak} \partial_{\alpha\dot{\alpha}} \bar{\psi}^{\dot{\alpha}k}}{1 + 4I\bar{\phi}} \right]. \end{aligned} \quad (4.88)$$

Let us now turn to the full  $\mathcal{N}=(1,0)$  supersymmetric gauge model which is defined by the sum of the classical actions (4.57) and (4.81). In this model one can perform the further change of fields of the vector multiplet in order to bring the total action  $S_{SYM} + S_{ad}$  to the simplest form,

$$\begin{aligned} \varphi &\rightarrow \hat{\varphi} = (1 + 4I\bar{\phi})^2 \varphi - \frac{4I(f_0^{ak} f_{0ak})}{1 + 4I\bar{\phi}}, \\ \psi_k^\alpha &\rightarrow \hat{\psi}_k^\alpha = (1 + 4I\bar{\phi})^2 \psi_k^\alpha - \frac{4I \rho_0^{ak} f_{0ak}}{1 + 4I\bar{\phi}}, \\ d_{kl} &\rightarrow \hat{d}_{kl} = (1 + 4I\bar{\phi}) d_{kl}. \end{aligned} \quad (4.89)$$

In terms of these new fields (4.89) the action  $S_{SYM} + S_{ad}$  reads

$$S_{SYM} + S_{ad} = \int d^4x (L_0 + L_{int}), \quad (4.90)$$

$$\begin{aligned} L_0 = & -\frac{1}{2}\hat{\varphi}\square\bar{\phi} + \frac{1}{2}\partial_m f_0^{ak}\partial_m f_{0ak} - \frac{1}{16}f^{\alpha\beta}f_{\alpha\beta} - i\hat{\psi}_k^\alpha\partial_{\alpha\dot{\alpha}}\bar{\psi}^{\dot{\alpha}k} \\ & + \frac{i}{2}\rho_0^{\alpha a}\partial_{\alpha\dot{\alpha}}\chi_{0a}^{\dot{\alpha}} + \frac{1}{4}\hat{d}^{kl}\hat{d}_{kl}, \end{aligned} \quad (4.91)$$

$$L_{int} = -\frac{1}{2}I\bar{\phi}(1 + 2I\bar{\phi})f^{\alpha\beta}f_{\alpha\beta} + \frac{I\rho_0^{\beta a}\rho_{0a}^\alpha f_{\alpha\beta}}{1 + 4I\bar{\phi}}. \quad (4.92)$$

Here  $f_{\alpha\beta} = i\partial_{\alpha\dot{\alpha}}a_{\dot{\beta}}^\alpha + i\partial_{\beta\dot{\alpha}}a_{\dot{\alpha}}^\alpha = (\sigma_{mn})_{\alpha\beta}f_{mn}$ . Note that  $L_0$  coincides with the Lagrangian of the free undeformed  $\mathcal{N}=(1,1)$  U(1) supergauge theory while  $L_{int}$  presents the interaction of hypermultiplet fields with the gauge multiplet. We see that the whole interaction is proportional to the deformation parameter  $I$ . Thereby, the interaction in this model is entirely an effect of chiral singlet deformation.

#### 4.4 Charged hypermultiplet model

Consider now the classical action (3.17) of the charged hypermultiplet model. Using the representation (3.21) and the explicit expression (3.20), this action can be written as

$$S_f[q^+, V^{++}] = \frac{1}{2} \int d\zeta du q_a^+ \left( D^{++} q^{+a} + \frac{1}{2}[V^{++} \star q^{+a}] - \frac{1}{2}(\tau_3)_b^a \{V^{++} \star q^{+b}\} \right). \quad (4.93)$$

The relevant superfield equation of motion reads

$$D^{++} q^{+a} + \frac{1}{2}[V^{++} \star q^{+a}] - \frac{1}{2}(\tau_3)_b^a \{V^{++} \star q^{+b}\} = 0. \quad (4.94)$$

To derive the component structure of the action (4.93), we follow the same steps as for the neutral hypermultiplet model considered in the previous subsection. We take the component expansions of the hypermultiplet (4.78) and the gauge superfield in the Wess-Zumino gauge (4.4) and substitute them into (4.94). As usual, the equations of motion for the auxiliary fields have the algebraic form and can be easily solved to eliminate these fields. As a result, we find the following component structure of the charged hypermultiplet superfield in terms of physical fields

$$\begin{aligned} q_e^{+a} = & u_k^+ f^{ak} + \theta^{+\alpha} \rho_\alpha^a + (\theta^+)^2 u_k^- g^{ak} + \bar{\theta}_\alpha^+ [\chi^{\dot{\alpha}a} + (\theta^+)^2 u_k^- u_l^- \sigma^{\dot{\alpha}a kl}] + \theta^+ \sigma_m \bar{\theta}^+ r_m^{ak} u_k^- \\ & + (\bar{\theta}^+)^2 [u_k^- h^{ak} + \theta^{+\alpha} u_k^- u_l^- \sigma_\alpha^{a kl} + (\theta^+)^2 u_k^- u_l^- u_j^- X^{a klj}], \end{aligned} \quad (4.95)$$

where

$$\begin{aligned} g^{ak} &= (\tau_3)_b^a \bar{\phi} f^{bk}, \quad r_m^{ak} = 2i(1 + 2I\bar{\phi})\partial_m f^{ak} + 2(\tau_3)_b^a A_m f^{bk}, \\ h^{ak} &= -4iI A_m \partial_m f^{ak} + (\tau_3)_b^a (\phi f^{bk} + 2I^2 \bar{\phi} \square f^{bk}), \\ \sigma^{\dot{\alpha}a kl} &= 2(\tau_3)_b^a \bar{\Psi}^{\dot{\alpha}(k} f^{bl)}, \quad \sigma_\alpha^{akl} = -4iI \bar{\psi}^{\dot{\alpha}(k} \partial_{\alpha\dot{\alpha}} f^{al)} + 2(\tau_3)_b^a \Psi_\alpha^{(k} f^{bl)}, \\ X^{a klj} &= (\tau_3)_b^a \mathcal{D}^{(kl} f^{bj)}. \end{aligned} \quad (4.96)$$

Now we substitute the expansions (4.95) and (4.4) into the action (4.93) and integrate over the Grassmann and harmonic variables to obtain the component form of the action of charged hypermultiplet in terms of the physical fields,

$$\begin{aligned}
S_f = & \int d^4x \left[ \frac{1}{2} (1 + 4I\bar{\phi}) \partial_m f_{ak} \partial_m f^{ak} + i(\tau_3)_a^b A_m f_{bk} \partial_m f^{ak} + \frac{1}{2} (A_m)^2 (f^{ak})^2 \right. \\
& + \frac{1}{2} \phi \bar{\phi} (f^{ak})^2 + I^2 (f^{ak})^2 \square (\bar{\phi}^2) - \frac{1}{2} (\tau_3)_b^a f_a^k f^{bl} \mathcal{D}_k l + 2iI \bar{\Psi}_k^{\dot{\alpha}} \rho_a^{\alpha} \partial_{\alpha\dot{\alpha}} f^{ak} \\
& + (\tau_3)_b^a \Psi_k^{\alpha} \rho_{\alpha a} f^{bk} + (\tau_3)_b^a f_a^k \bar{\psi}_{\dot{\alpha}k} \chi^{\dot{\alpha}b} + \frac{i}{2} (1 + 2I\bar{\phi}) \rho^{\alpha a} \partial_{\alpha\dot{\alpha}} \chi_a^{\dot{\alpha}} \\
& - \frac{1}{2} (\tau_3)_b^a \rho_a^{\alpha} A_{\alpha\dot{\alpha}} \chi^{\dot{\alpha}b} + \frac{1}{4} (\tau_3)_b^a (\bar{\phi} \chi_{\dot{\alpha}a} \chi^{\dot{\alpha}b} + \phi \rho_a^{\alpha} \rho_{\alpha}^b) + iI \rho^{\alpha a} A_m \partial_m \rho_{\alpha a} \\
& \left. + \frac{i}{2} I \rho^{\beta a} \rho_a^{\alpha} \partial_{(\alpha\dot{\alpha}} A_{\beta)}^{\dot{\alpha}} + I^2 (\tau_3)_b^a \bar{\phi} \partial_{\alpha\dot{\alpha}} \rho_{\beta a} \partial^{\beta\dot{\alpha}} \rho^{\alpha b} \right]. \quad (4.97)
\end{aligned}$$

Note that in the limit  $I \rightarrow 0$  the action (4.97) still retains an interaction. It has the standard form of the interaction between the physical fields of  $\mathcal{N}=(1, 1)$  supersymmetric electrodynamics.

As in the neutral hypermultiplet model, from (3.15) one can derive the residual gauge transformations for the physical component fields (in the Wess-Zumino gauge for the vector multiplet),

$$\delta_r f^{ak} = i\lambda (\tau_3)_b^a f^{bk}, \quad \delta_r \rho_a^{\alpha} = i\lambda (\tau_3)_b^a \rho_{\alpha}^b, \quad \delta_r \chi^{\dot{\alpha}a} = i\lambda (\tau_3)_b^a \chi^{\dot{\alpha}b} - 2I \partial^{\alpha\dot{\alpha}} \lambda \rho_{\alpha}^a. \quad (4.98)$$

The  $\mathcal{N}=(1, 0)$  supersymmetry transformations for these fields are given by

$$\delta_{\epsilon} f^{ak} = \epsilon^{\alpha k} \rho_{\alpha}^a, \quad \delta_{\epsilon} \rho_{\alpha}^a = 2\epsilon_{\alpha}^k (\tau_3)_b^a \bar{\phi} f_k^b, \quad \delta_{\epsilon} \chi_{\dot{\alpha}}^a = -2\epsilon_k^{\alpha} [i(1 + 2I\bar{\phi}) \partial_{\alpha\dot{\alpha}} f^{ak} + (\tau_3)_b^a A_{\alpha\dot{\alpha}} f^{bk}]. \quad (4.99)$$

It is easy to check that the action (4.97) is invariant under both the gauge transformations (4.98) and the supersymmetry ones (4.99).

In the charged hypermultiplet model there also exists a field redefinition (Seiberg-Witten map) which casts the transformations (4.98), (4.99) in the undeformed form,

$$\begin{aligned}
f^{ak} & \rightarrow f_0^{ak} = (1 + 2I\bar{\phi}) f^{ak}, \\
\rho^{\alpha a} & \rightarrow \rho_0^{\alpha a} = (1 + 2I\bar{\phi}) \rho^{\alpha a}, \\
\chi_{\dot{\alpha}}^a & \rightarrow \chi_{\dot{\alpha}0}^a = \chi_{\dot{\alpha}}^a - \frac{2I A_{\alpha\dot{\alpha}} \rho^{\alpha a}}{1 + 4I\bar{\phi}} + \frac{4I \bar{\Psi}_{\dot{\alpha}k} f^{ak}}{1 + 4I\bar{\phi}}. \quad (4.100)
\end{aligned}$$

However, in contrast to the neutral hypermultiplet model, the map (4.100) does not lead to the substantial simplifications of the classical action. Therefore, here we do not give how the classical action of charged hypermultiplet looks in terms of the new fields  $f_0^{ak}$ ,  $\rho_0^{\alpha a}$ ,  $\chi_{\dot{\alpha}0}^a$ .

## 5 Renormalizability of the $\mathcal{N}=(1, 0)$ non-anticommutative abelian models

In this section we explore the quantum aspects of the non-anticommutative theories defined by the classical actions (4.38), (4.81) and (4.93).

From the point of view of physical applications only renormalizable theories play a fundamental role in quantum field theory while the non-renormalizable ones are usually treated as some effective theories. By the term “renormalizability” we will mean the multiplicative renormalizability, when all the divergent quantum corrections in a given theory have the form of some terms of the classical action and hence can be taken away by some redefinition of coupling constants or fields in the classical action. According to the customary lore of quantum field theory, a model is power-counting non-renormalizable if it involves coupling constants of the negative mass dimension. The supersymmetric models with the chiral singlet deformation under consideration contain the parameter of non-anticommutativity  $I$  with the negative mass dimension,  $[I] = -1$ . If one treats this parameter as a coupling constant, the considered models are formally non-renormalizable. Nevertheless, we will show that in our case the standard arguments towards non-renormalizability fail and all the models considered here are renormalizable. A key feature of the non-anticommutativity is that all such models are formulated only in the Euclidean superspace and the deformation is present only in the chiral sector of superspace, while the antichiral one remains intact. Therefore, the interaction terms in the actions appear in a non-symmetric way, still preserving the reality with respect to the conjugation (2.11) in the Euclidean space. These interactions lead to the quantum divergences of a special form which do not violate the renormalizability. As a result, the non-anticommutative theories with  $\mathcal{N}=(1,0)$  supersymmetry are renormalizable and so they can bear certain interest for the further study in the framework of quantum field theory.

Here we will prove the renormalizability of the models with the classical actions (4.38), (4.81), (4.93). For this purpose we will calculate the divergent parts of the effective actions of these models. By definition, the effective action in quantum field theory is a generating functional of all connected one-particle irreducible Green functions. It encodes the full information about the quantum dynamics of the given field theory and, in particular, allows one to find the structure of quantum divergences. To obtain the divergent parts of the effective actions we employ here the standard methods of quantum field theory based on the Feynman diagram techniques.

## 5.1 Gauge superfield model

Consider the non-anticommutative model of abelian gauge superfield in its component formulation with the classical action (4.38). As a first step, we eliminate the auxiliary field  $\mathcal{D}^{ij}$  by its equation of motion,

$$\mathcal{D}^{ij} = -\frac{8I\bar{\Psi}_{\dot{\alpha}}^i\bar{\Psi}^{j\dot{\alpha}}}{1+4I\bar{\phi}}. \quad (5.1)$$

Upon substituting (5.1) into (4.40), the action for the spinor fields takes the form

$$S_{\Psi} = i \int d^4x \left( \Psi^{i\alpha} + \frac{4IA_m\sigma_m^{\alpha\dot{\alpha}}\bar{\Psi}^{i\dot{\alpha}}}{1+4I\bar{\phi}} \right) (\sigma_n)_{\alpha\dot{\beta}} \partial_n \left( \frac{\bar{\Psi}_i^{\dot{\beta}}}{1+4I\bar{\phi}} \right). \quad (5.2)$$

In what follows we will consider the quantization of the model (4.38) with the action  $S_{\Psi}$  given by (5.2).

Since the action (4.38) is invariant under the gauge transformations (4.7) one needs to fix the gauge to quantize the theory. It is convenient to choose the following gauge-fixing condition

$$\partial_m \frac{A_m}{1 + 4I\bar{\phi}} = 0. \quad (5.3)$$

Note that (5.3) is none other than the Lorentz gauge condition  $\partial_m a_m = 0$  for the gauge field  $a_m = A_m/(1 + 4I\bar{\phi})$  which transforms in a standard way under the U(1) gauge group,  $\delta a_m = \partial_m \lambda$ .

Further we follow the routine of Faddeev-Popov procedure to fix the gauge freedom in the functional integral. Let us introduce the corresponding gauge-fixing function

$$\chi = \partial_m \frac{A_m}{1 + 4I\bar{\phi}} = \frac{\partial_m A_m - A_m G_m}{1 + 4I\bar{\phi}}, \quad (5.4)$$

where

$$G_m(x) = \partial_m \ln[1 + 4I\bar{\phi}(x)]. \quad (5.5)$$

The function  $\chi$  transforms under gauge transformations (4.7) as follows

$$\delta\chi = \partial_m \frac{\delta A_m}{1 + 4I\bar{\phi}} = \square\lambda. \quad (5.6)$$

The relation (5.6) shows that the action for the ghost fields is just the action of free scalars

$$S_{FP} = \int d^4x b \square c. \quad (5.7)$$

The generating functional for the Green's functions is now defined as

$$Z[J] = \int \mathcal{D}(\phi, \bar{\phi}, \Psi, \bar{\Psi}, A_m, b, c) \delta(\chi - \frac{\partial_m A_m - A_m G_m}{1 + 4I\bar{\phi}}) e^{-\frac{1}{2}(S_{SYM} + S_{FP} + S_J)}, \quad (5.8)$$

where

$$S_J = \int d^4x [\phi J_\phi + \bar{\phi} J_{\bar{\phi}} + \Psi_\alpha^i (J_\Psi)_i^\alpha + \bar{\Psi}_{i\dot{\alpha}} (J_{\bar{\Psi}})^{i\dot{\alpha}} + A_m (J_A)_m] \quad (5.9)$$

and  $J_\phi, J_{\bar{\phi}}, (J_\Psi)_i^\alpha, (J_{\bar{\Psi}})^{i\dot{\alpha}}, (J_A)_m$  are sources for the fields  $\phi, \bar{\phi}, \Psi_\alpha^i, \bar{\Psi}_{i\dot{\alpha}}, A_m$ . We have inserted into (5.8) the functional delta-function that fixes the gauge degrees of freedom in the functional integral over the gauge fields. This delta-function can be easily written in the Gaussian form by averaging (5.8) with the factor

$$1 = \int \mathcal{D}\chi e^{-\frac{\alpha}{2} \int d^4x \chi^2 (1 + 4I\bar{\phi})^2} = \text{Det}^{-1/2} [\delta^4(x - x') (1 + 4I\bar{\phi})^2]. \quad (5.10)$$

The functional integral (5.10) produces the following gauge-fixing action

$$\begin{aligned} S_{gf} &= \frac{\alpha}{2} \int d^4x (\partial_m A_m - A_m G_m)^2 \\ &= \frac{\alpha}{2} \int d^4x [(\partial_m A_m)^2 - 2\partial_m A_m A_n G_n + A_m A_n G_m G_n]. \end{aligned} \quad (5.11)$$



Here  $\alpha$  is an arbitrary parameter. For simplicity, in the sequel we set  $\alpha = 1$ . As a result, the generating functional (5.8) can be represented in the following form

$$Z[J] = \int \mathcal{D}(\phi, \bar{\phi}, \Psi, \bar{\Psi}, A_m, b, c) e^{-\frac{1}{2}(S_{\text{tot}} + S_{FP} + S_J)}, \quad (5.12)$$

where

$$\begin{aligned} S_{\text{tot}} &= S_{SYM} + S_{gf} \\ &= -\frac{1}{2} \int d^4x \square \bar{\phi} (\phi + 4I^2 \partial_m \bar{\phi} G_m) \\ &\quad + i \int d^4x \left( \Psi^{i\alpha} + \frac{4IA_m \sigma_m^{\alpha\dot{\alpha}} \bar{\Psi}^{i\dot{\alpha}}}{1 + 4I\bar{\phi}} \right) (\sigma_n)_{\alpha\dot{\beta}} \partial_n \left( \frac{\bar{\Psi}_i^{\dot{\beta}}}{1 + 4I\bar{\phi}} \right) \\ &\quad - \int d^4x \left[ \frac{1}{2} A_n \square A_n - A_n G_m \partial_n A_m + A_n G_n \partial_m A_m + \varepsilon_{mnrs} G_m A_n \partial_r A_s \right]. \end{aligned} \quad (5.13)$$

The functional integral (5.12) with the action (5.13) requires several comments.

1. The ghost fields  $b, c$  enter the action only through their kinetic term. Hence, they fully decouple and can be integrated out.
2. The fermionic fields  $\Psi_\alpha^i, \bar{\Psi}_\alpha^i$  do not contribute to the effective action. Indeed, the action  $S_\Psi$  (5.2) can be brought to the form of free action by the following change of fields  $\psi^{i\alpha} = \Psi^{i\alpha} + \frac{4IA_m \sigma_m^{\alpha\dot{\alpha}} \bar{\Psi}^{i\dot{\alpha}}}{1 + 4I\bar{\phi}}, \bar{\psi}_i^{\dot{\beta}} = \frac{\bar{\Psi}_i^{\dot{\beta}}}{1 + 4I\bar{\phi}}$ . One can also check this observation by the direct computations of the corresponding Feynman diagrams.
3. The contribution to the effective action from the scalar field  $\phi$  is also trivial since it appears in the action (5.13) without interaction with other fields.
4. A non-trivial contribution to the effective action in this model comes only from the terms in the last line of (5.13). These terms are quadratic in the vector field  $A_m$  and linear with respect to  $G_m$ . Hence, the field  $G_m$  appears only on the external lines while  $A_m$  works only inside the Feynman diagrams. Moreover, there are only one-loop diagrams since there are no self-interaction of  $A_m$ . Since the field  $G_m$  is expressed only through  $\bar{\phi}$  as in (5.5), we conclude that the effective action is a functional of  $\bar{\phi}$  only. The dimensional considerations allow one to construct only the following three terms in the effective action

$$\Gamma = \int d^4x [f_1(I\bar{\phi}) I^2 \square \bar{\phi} \square \bar{\phi} + f_2(I\bar{\phi}) I^3 \square \bar{\phi} \partial_m \bar{\phi} \partial_m \bar{\phi} + f_3(I\bar{\phi}) I^4 (\partial_m \bar{\phi} \partial_m \bar{\phi})^2], \quad (5.14)$$

where  $f_1, f_2, f_3$  are some functions. The Feynman graph computations should specify these functions.

Taking into account these comments, we conclude that the effective action in this model can be represented by the following formal expression <sup>6</sup>

$$\Gamma^{SYM} = \frac{1}{2} \text{Tr} \ln \frac{\delta^2 \tilde{S}}{\delta A_p(x) \delta A_q(x')}, \quad (5.15)$$

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<sup>6</sup>Note that the one-loop effective action in the Euclidean space is given by  $\Gamma = \frac{1}{2} \text{Tr} \ln S''[\Phi]$  rather than the Minkowski space expression  $\Gamma = \frac{i}{2} \text{Tr} \ln S''[\Phi]$ . Here  $S''[\Phi]$  is the second functional derivative of the classical action.

where  $\tilde{S}$  is the last line in (5.13),

$$\tilde{S} = \int d^4x \left[ -\frac{1}{2} A_n \square A_n + A_n G_m \partial_n A_m - A_n G_n \partial_m A_m - \varepsilon_{mnrs} G_m A_n \partial_r A_s \right]. \quad (5.16)$$

The second functional derivative of the action (5.16) can be easily calculated,

$$\frac{\delta^2 \tilde{S}}{\delta A_p(x) \delta A_q(x')} = -\delta_{pq} \square \delta^4(x - x') + 4G_{[q} \partial_{p]} \delta^4(x - x') + 2\varepsilon_{pqmn} G^m \partial^n \delta^4(x - x'). \quad (5.17)$$

Substituting (5.17) into (5.15) we have

$$\begin{aligned} \Gamma^{SYM} &= \frac{1}{2} \text{Tr} \ln [\delta_{pq} \square \delta^4(x - x') + 4G_{[p} \partial_{q]} \frac{1}{\square} \delta^4(x - x') - 2\varepsilon_{pqmn} G_m \frac{\partial_n}{\square} \delta^4(x - x')] \\ &= \frac{1}{2} \text{Tr} \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} [4G_{[p} \partial_{q]} \frac{1}{\square} \delta^4(x - x') - 2\varepsilon_{pqmn} G_m \frac{\partial_n}{\square} \delta^4(x - x')]^j. \end{aligned} \quad (5.18)$$

The expression (5.18) provides us with the perturbative expansion of the effective action in a form of Feynman diagram series with the external lines  $G_m$ .

The propagators in (5.18) appear in the combination  $\partial_m \square^{-1} \delta^4(x - x')$ . On the dimensionality grounds, only the expressions like

$$\left[ \frac{\partial_m}{\square} \delta^4(x - x') \right]^2, \quad \left[ \frac{\partial_m}{\square} \delta^4(x - x') \right]^3, \quad \left[ \frac{\partial_m}{\square} \delta^4(x - x') \right]^4 \quad (5.19)$$

are divergent and all higher powers of  $\partial_m \square^{-1} \delta^4(x - x')$  produce finite contributions to the effective action. Therefore, only two- three- and four-point diagrams lead to the divergent contributions in the effective action (note that the external line is that of the field  $G_m$ ). We are interested solely in the divergent contributions to the effective action, and consider the calculations of two-, three- and four-point functions separately.

Let us consider only the terms in the series (5.18) which are responsible for the two-, three- and four-point diagram contributions,

$$\begin{aligned} \Gamma_2^{SYM} &= - \int d^4x_1 d^4x_2 [2G_{[q}(x_1) \partial_{p]} \frac{1}{\square} \delta^4(x_1 - x_2) + \varepsilon_{pqmn} G_m(x_1) \partial_n \frac{1}{\square} \delta^4(x_1 - x_2)] \\ &\quad \times [2G_{[p}(x_2) \partial_{q]} \frac{1}{\square} \delta^4(x_2 - x_1) + \varepsilon_{qprs} G_r(x_2) \partial_s \frac{1}{\square} \delta^4(x_2 - x_1)], \end{aligned} \quad (5.20)$$

$$\begin{aligned} \Gamma_3^{SYM} &= \frac{4}{3} \int d^4x_1 d^4x_2 d^4x_3 [(G_t(x_1) \partial_u - G_u(x_1) \partial_t - \varepsilon_{tumn} G_m(x_1) \partial_n) \frac{1}{\square} \delta^4(x_1 - x_2)] \\ &\quad \times [(G_u(x_2) \partial_w - G_w(x_2) \partial_u - \varepsilon_{uwrs} G_r(x_2) \partial_s) \frac{1}{\square} \delta^4(x_2 - x_3)] \\ &\quad \times [(G_w(x_3) \partial_t - G_t(x_3) \partial_w - \varepsilon_{wtpq} G_p(x_3) \partial_q) \frac{1}{\square} \delta^4(x_3 - x_1)], \end{aligned} \quad (5.21)$$

$$\begin{aligned} \Gamma_4^{SYM} &= -2 \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 [(G_p \partial_q - G_q \partial_p - \varepsilon_{pqqr'} G_{p'} \partial_{q'}) \frac{1}{\square} \delta^4(x_1 - x_2)] \\ &\quad \times [(G_q \partial_m - G_m \partial_q - \varepsilon_{m'mn'q} G_{m'} \partial_{n'}) \frac{1}{\square} \delta^4(x_2 - x_3)] \\ &\quad \times [(G_m \partial_n - G_n \partial_m - \varepsilon_{r'ns'm} G_{r'} \partial_{s'}) \frac{1}{\square} \delta^4(x_3 - x_4)] \\ &\quad \times [(G_n \partial_p - G_p \partial_n - \varepsilon_{tpun} G_t \partial_u) \frac{1}{\square} \delta^4(x_4 - x_1)]. \end{aligned} \quad (5.22)$$

To proceed, one has to perform the integrations over  $x_2, x_3, x_4$  in the expressions (5.20), (5.21), (5.22) using the Fourier representation for the fields and delta-functions. The corresponding divergent momentum integrals should be regularized by the standard methods of quantum field theory using, e.g., the dimensional regularization. Here we omit the details of these computations which can be found in [32]. As a result, the divergent parts of the functions (5.20), (5.21), (5.22) are given by

$$\Gamma_{2,div}^{SYM} = \frac{1}{16\pi^2\varepsilon} \int d^4x \ln(1 + 4I\bar{\phi}) \square^2 \ln(1 + 4I\bar{\phi}), \quad (5.23)$$

$$\Gamma_{3,div}^{SYM} = -\frac{1}{4\pi^2\varepsilon} \int d^4x \partial_m \ln(1 + 4I\bar{\phi}) \partial_m \ln(1 + 4I\bar{\phi}) \square \ln(1 + 4I\bar{\phi}), \quad (5.24)$$

$$\Gamma_{4,div}^{SYM} = -\frac{5}{16\pi^2\varepsilon} \int d^4x [\partial_m \ln(1 + 4I\bar{\phi}) \partial_m \ln(1 + 4I\bar{\phi})]^2. \quad (5.25)$$

Here  $\varepsilon$  is a parameter of dimensional regularization,  $\varepsilon = 2 - d/2$ , where  $d$  is the dimension of space-time. The limit  $\varepsilon \rightarrow 0$  takes off the regularization. The full divergent contribution to the effective action is given by the sum of (5.23), (5.24) and (5.25). Using the integration by parts, the divergent contribution to the effective action can be brought to the form

$$\Gamma_{div}^{SYM} = \Gamma_{2,div}^{SYM} + \Gamma_{3,div}^{SYM} + \Gamma_{4,div}^{SYM} = \frac{1}{\pi^2\varepsilon} \int d^4x \frac{I^2 \square \bar{\phi} \square \bar{\phi}}{(1 + 4I\bar{\phi})^2} - \frac{6}{\pi^2\varepsilon} \int d^4x \frac{4I^3 \square \bar{\phi} \partial_m \bar{\phi} \partial_m \bar{\phi}}{(1 + 4I\bar{\phi})^3}. \quad (5.26)$$

Note that the action (5.26) matches with the previously guessed structure (5.14).

At first sight, the non-anticommutative supergauge model looks non-renormalizable, since the quantum computations produce the expressions (5.26) which are absent in the classical action (4.38). However, it is easy to see that the divergent terms (5.26) being added to the classical action (4.38) can be completely compensated by the following shift of scalar field  $\phi$

$$\phi \longrightarrow \phi - \frac{2}{\pi^2\varepsilon} \frac{I^2 \square \bar{\phi}}{(1 + 4I\bar{\phi})^2} + \frac{12}{\pi^2\varepsilon} \frac{4I^3 \partial_m \bar{\phi} \partial_m \bar{\phi}}{(1 + 4I\bar{\phi})^3}. \quad (5.27)$$

Therefore, the  $\mathcal{N}=(1,0)$  gauge model is renormalizable in the sense that all divergences can be removed by the redefinition of the scalar field  $\phi$ . One can treat (5.27) as a change of fields in the functional integral (5.12). Since the Jacobian of such a change of functional variables is equal to unity, the terms (5.26), being added to the classical action (4.38), do not contribute to the effective action. Moreover, this model is finite since the shift (5.27) allows one to completely eliminate the divergences from the effective action.

This situation is analogous to the  $\mathcal{N}=(1/2,0)$  SYM model considered in [19], where it was demonstrated that the quantum computations in this model generate the divergent terms which are not present in the classical action of the model, but these extra divergences can be removed by a simple shift of the gaugino field (the lowest component in  $\mathcal{N}=(1/2,1/2)$  gauge multiplet). In our case the divergences can also be removed by the shift of lowest component of  $\mathcal{N}=(1,1)$  gauge multiplet (scalar field).

It should also be noted that the divergent expression (5.26) vanishes on the classical equation of motion for the scalar field  $\bar{\phi}$  given by  $\square \bar{\phi} = 0$ . Therefore the  $S$ -matrix in this model is free of divergences and in this sense one can say that the model under consideration is finite.

## 5.2 Neutral hypermultiplet model

Consider the model of neutral hypermultiplet with the classical action (4.81). Clearly, the hypermultiplet fields  $f^{ak}$ ,  $\rho^{\alpha a}$ ,  $\chi_a^{\dot{\alpha}}$  work only inside the loops of Feynman diagrams while the vector multiplet fields  $\bar{\phi}$ ,  $\bar{\Psi}_k^{\dot{\alpha}}$ ,  $A_m$  appear only on external lines. Moreover, since the action (4.81) is quadratic with respect to the hypermultiplet fields, the corresponding effective action is one-loop exact. It is easy to observe also that the terms in the second line of the action (4.81) correspond to the interaction vertices which do not couple with the other vertices in one-loop diagrams. Indeed, to form a loop with these vertices one needs the propagators  $\langle \rho^{\alpha a} f^{bk} \rangle$ ,  $\langle \rho^{\alpha a} \rho_{\beta b} \rangle$  which are absent in this model.

Let us analyze also the term  $\frac{i}{2}(1 + 4I\bar{\phi})\rho^{\alpha a}\partial_{\alpha\dot{\alpha}}\chi_a^{\dot{\alpha}}$  in the first line of (4.81). It is easy to see that this term does not contribute to the effective action,

$$\begin{aligned}\Gamma_{ferm} &= -\text{Tr} \ln \frac{\delta^2 S_{ad}}{\delta \rho^{\alpha a}(x) \delta \chi_b^{\dot{\alpha}}(x')} = -\text{Tr} \ln \left[ \frac{i}{2}(1 + 4I\bar{\phi})\partial_{\alpha\dot{\alpha}}\delta^4(x - x')\delta_a^b \right] \\ &= -2\text{Tr} \ln \left[ \frac{i}{2}(1 + 4I\bar{\phi})\delta^4(x - x') \right] - 2\text{Tr} \ln [\partial_{\alpha\dot{\alpha}}\delta^4(x - x')] \simeq 0.\end{aligned}\quad (5.28)$$

As a result, the non-trivial contribution to the effective action comes only from the loops with the internal lines given by the scalar fields  $f^{ak}$  and with the field  $\bar{\phi}$  on external lines. This contribution is given by the following formal expression

$$\Gamma^{hyp} = \frac{1}{2}\text{Tr} \ln \frac{\delta^2}{\delta f^{ak}(x) \delta f_{a'k'}(x')} \left[ \frac{1}{2} \int d^4x (1 + 4I\bar{\phi})^2 \partial_m f^{ak} \partial_m f_{ak} \right]. \quad (5.29)$$

Calculating the variational derivative in (5.29), we obtain

$$\Gamma^{hyp} = 2\text{Tr} \ln \left[ (1 + 4I\bar{\phi})^2 \square \delta^4(x - x') \right] + 2\text{Tr} \ln \left[ \delta^4(x - x') + 2 \frac{1}{\square} G_m(x) \partial_m \delta^4(x - x') \right]. \quad (5.30)$$

The first term in the r.h.s. of (5.30) is trivial since it is proportional to  $\delta^4(0)$  that is zero in the sense of dimensional regularization. The second term in the r.h.s. of (5.30) provides us with the following perturbative representation for the effective action

$$\Gamma^{hyp} = 2\text{Tr} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left[ \frac{2}{\square} G_m(x) \partial_m \delta^4(x - x') \right]^n. \quad (5.31)$$

Note that the fields  $G_m$  in the series (5.31) play the role of external lines of corresponding Feynman diagrams. Taking into account the dimensions of field  $G_m$  and propagators  $\frac{1}{\square} \partial_m \delta^4(x - x')$  we conclude that only the diagrams with two, three and four external lines are divergent. Let us consider these divergent terms in the series (5.31), which corresponds to  $n = 2, 3, 4$ :

$$\Gamma_2^{hyp} = -4 \int d^4x_1 d^4x_2 G_m(x_1) G_n(x_2) \frac{\partial_m}{\square} \delta^4(x_1 - x_2) \frac{\partial_n}{\square} \delta^4(x_2 - x_1), \quad (5.32)$$

$$\begin{aligned}\Gamma_3^{hyp} &= \frac{16}{3} \int d^4x_1 d^4x_2 d^4x_3 G_m(x_1) G_n(x_2) G_p(x_3) \\ &\quad \times \frac{\partial_m}{\square} \delta^4(x_1 - x_2) \frac{\partial_n}{\square} \delta^4(x_2 - x_3) \frac{\partial_p}{\square} \delta^4(x_3 - x_1),\end{aligned}\quad (5.33)$$

$$\begin{aligned}\Gamma_4^{hyp} &= -8 \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 G_m(x_1) G_n(x_2) G_p(x_3) G_r(x_4) \\ &\quad \times \frac{\partial_m}{\square} \delta^4(x_1 - x_2) \frac{\partial_n}{\square} \delta^4(x_2 - x_3) \frac{\partial_p}{\square} \delta^4(x_3 - x_4) \frac{\partial_r}{\square} \delta^4(x_4 - x_1).\end{aligned}\quad (5.34)$$

The expressions (5.32), (5.33), (5.34) can be calculated by the standard methods of quantum field theory, see [32] for details. Here we give only the results,

$$\Gamma_{2,div}^{hyp} = -\frac{1}{16\pi^2\varepsilon} \int d^4x \ln(1 + 4I\bar{\phi}(x)) \square^2 \ln(1 + 4I\bar{\phi}(x)), \quad (5.35)$$

$$\Gamma_{3,div}^{hyp} = -\frac{1}{8\pi^2\varepsilon} \int d^4x \square \ln(1 + 4I\bar{\phi}) \partial_n \ln(1 + 4I\bar{\phi}) \partial_n \ln(1 + 4I\bar{\phi}), \quad (5.36)$$

$$\Gamma_{4,div}^{hyp} = -\frac{1}{16\pi^2\varepsilon} \int d^4x [\partial_m \ln(1 + 4I\bar{\phi}) \partial_m \ln(1 + 4I\bar{\phi})]^2. \quad (5.37)$$

Summarizing (5.35), (5.36), (5.37) we obtain the full divergent contribution to the hypermultiplet effective action,

$$\Gamma_{div}^{hyp} = -\frac{1}{\pi^2\varepsilon} \int d^4x \frac{I^2 \square \bar{\phi} \square \bar{\phi}}{(1 + 4I\bar{\phi})^2}. \quad (5.38)$$

Note that (5.38) agrees with the general structure of the effective action (5.14) guessed before.

As in the deformed gauge model, there are no terms in the classical action (4.38) having the field structure similar to (5.38). Therefore, at first sight the multiplicative renormalizability of the model can be spoiled by the divergent contribution (5.38). However, it is easy to observe that the term (5.38), being added to the classical action (4.38), can be completely compensated by the following shift of scalar field

$$\phi \longrightarrow \phi + \frac{2}{\pi^2\varepsilon} \frac{I^2 \square \bar{\phi}}{(1 + 4I\bar{\phi})^2}, \quad (5.39)$$

while the other fields remain intact. Since the Jacobian of the change (5.39) is equal to unity, we conclude that the term (5.38) does not spoil the renormalizability and finiteness of the model. Moreover, the divergent term (5.38) vanishes on the classical equation of motion for the scalar field  $\bar{\phi}$  given by  $\square \bar{\phi} = 0$ . Therefore the finiteness of  $S$ -matrix is also evident.

Let us consider finally the general abelian  $\mathcal{N}=(1,0)$  non-anticommutative model of gauge superfield interacting with the hypermultiplet matter. It is described by the classical action

$$S = S_{SYM} + S_{ad}, \quad (5.40)$$

where  $S_{SYM}$  and  $S_{hyp}$  are given by (4.38), (4.81), respectively. It is easy to see that the divergent part of the effective action in this model is given by the sum of the expressions (5.26) and (5.38),

$$\Gamma_{div} = \Gamma_{div}^{SYM} + \Gamma_{div}^{hyp} = -\frac{6}{\pi^2\varepsilon} \int d^4x \frac{4I^3 \square \bar{\phi} \partial_m \bar{\phi} \partial_m \bar{\phi}}{(1 + 4I\bar{\phi})^3}. \quad (5.41)$$

The divergent expression (5.41) can also be completely compensated by the following shift of the scalar  $\phi$ ,

$$\phi \longrightarrow \phi + \frac{12}{\pi^2\varepsilon} \frac{4I^3 \partial_m \bar{\phi} \partial_m \bar{\phi}}{(1 + 4I\bar{\phi})^3}. \quad (5.42)$$

As a result, the general  $\mathcal{N}=(1,0)$  supergauge theory is also renormalizable and finite.

### 5.3 Charged hypermultiplet model

The model of charged hypermultiplet is described by the superfield action (3.14) or the corresponding component field action (4.97). Note that the component action (4.97) is much more complicated than the actions of neutral hypermultiplet and gauge superfield considered above. Therefore, to prove the renormalizability of the charged hypermultiplet model we prefer to use the superfield description (3.14).

In superfields, the effective action of charged hypermultiplet is given by the following formal expression

$$\Gamma = \text{Tr} \ln \frac{\delta^2 S_f}{\delta \tilde{q}^+(1) \delta q^+(2)} = \text{Tr} \ln (D^{++} + V^{++} \star). \quad (5.43)$$

The free Green function of hypermultiplet superfield has the standard form [35],

$$G_0^{(1,1)}(1|2) = -\frac{1}{\square} (D_1^+)^4 (D_2^+)^4 \frac{\delta^{12}(z_1 - z_2)}{(u_1^+ u_2^+)^3}. \quad (5.44)$$

It solves free equation of motion with the delta-source,  $D^{++} G_0^{(1,1)}(1|2) = \delta_A^{(1,3)}(1|2)$ , where  $\delta_A^{(1,3)}(1|2)$  is an analytic delta-function (see [35] for the details of the harmonic superspace approach). The effective action (5.43) can be formally expressed through the free Green function (5.44) as

$$\Gamma = \text{Tr} \ln [\delta_A^{(3,1)}(1|2) + V^{++}(1) \star G_0^{(1,1)}(1|2)]. \quad (5.45)$$

Expanding the logarithm in (5.45) in a series, we obtain a perturbative representation for the effective action,

$$\Gamma = \sum_{n=2}^{\infty} \Gamma_n, \quad (5.46)$$

$$\Gamma_n = \frac{(-1)^{n+1}}{n} \int d\zeta_1 du_1 \dots d\zeta_n du_n V^{++}(1) \star G_0^{(1,1)}(1|2) \dots V^{++}(n) \star G_0^{(1,1)}(n|1). \quad (5.47)$$

We calculate further the divergent parts of the  $n$ -point functions  $\Gamma_n$ .

As the first step we restore in (5.47) the full  $\mathcal{N}=(1,1)$  superspace integration measure with the help of  $(D^+)^4$  factors of the propagator (5.44). For this purpose we apply the following standard identity

$$d^{12}z = d^4x d^8\theta = d\zeta (D^+)^4. \quad (5.48)$$

As a result, the  $n$ -point Green function reads

$$\begin{aligned} \Gamma_n = & -\frac{1}{n} \int d^{12}z_1 du_1 \dots d^{12}z_n du_n V^{++}(1) \star \frac{1}{\square} (D_1^+)^4 \frac{\delta^{12}(z_1 - z_2)}{(u_1^+ u_2^+)^3} \\ & \times V^{++}(2) \star \frac{1}{\square} (D_2^+)^4 \frac{\delta^{12}(z_2 - z_3)}{(u_2^+ u_3^+)^3} \dots V^{++}(n) \star \frac{1}{\square} (D_n^+)^4 \frac{\delta^{12}(z_n - z_1)}{(u_n^+ u_1^+)^3}. \end{aligned} \quad (5.49)$$

Further we integrate by parts and take off the integration over Grassmann variables  $\theta_2, \dots, \theta_{n-1}$  using the corresponding delta-functions. As a consequence, for the expression (5.49) we have

$$\Gamma_n = -\frac{1}{n} \int d^{12}z_1 d^{12}z_n d^4x_2 \dots d^4x_{n-1} d^n U X_n[U] \delta^8(\theta_1 - \theta_n)$$

$$\begin{aligned}
& \times V^{++}(n) \star (D_{n-1}^+)^4 [(D_{n-2}^+)^4 [\dots \\
& (D_2^+)^4 [(D_1^+)^4 (D_n^+)^4 \frac{1}{\square} \delta^{12}(z_n - z_1) \star V^{++}(1) \frac{1}{\square} \delta^4(x_1 - x_2)] \\
& \star V^{++}(x_2, \theta_1, u_2) \frac{1}{\square} \delta^4(x_2 - x_3)] \dots \star V^{++}(x_{n-2}, \theta_1, u_{n-2}) \frac{1}{\square} \delta^4(x_{n-2} - x_{n-1})] \\
& \star V^{++}(x_{n-1}, \theta_1, u_{n-1}) \frac{1}{\square} \delta^4(x_{n-1} - x_n). \tag{5.50}
\end{aligned}$$

Here we have introduced the denotations  $d^n U = du_1 \dots du_n$ ,  $X_n[U] = 1/[(u_1^+ u_2^+)^3 \dots (u_n^+ u_1^+)^3]$ . Note that the covariant derivatives in (5.50) commute with the  $\star$ -product operators and hit only the corresponding  $V^{++}$  superfields and delta-functions.

To integrate over the remaining Grassmann variables in (5.50) we make the Fourier transform for superfields and delta-functions,

$$V^{++}(1) = \int \frac{d^4 p_1}{(2\pi)^4} d^4 \rho_1 e^{ip_1 x_1} e^{\rho_1 \theta_1} \tilde{V}^{++}(p_1, \rho_1, \bar{\theta}_1, u_1), \tag{5.51}$$

$$\delta^{12}(z_1 - z_2) = \int \frac{d^4 k_1}{(2\pi)^4} d^4 \pi_1 e^{ik_1(x_1 - x_2)} e^{\pi_1(\theta_1 - \theta_2)} \delta^4(\bar{\theta}_1 - \bar{\theta}_2), \tag{5.52}$$

where we denote  $\pi\theta = \pi_i^\alpha \theta_\alpha^i = -\theta_i^\alpha \pi_\alpha^i = -\theta\pi$ . The  $\star$ -product of Fourier transforms of arbitrary two superfunctions  $f, g$  is given by

$$f \star g = \int \frac{d^4 p d^4 k}{(2\pi)^8} d^4 \rho d^4 \pi \tilde{f}(p, \rho) \tilde{g}(k, \pi) e^{i(p+k)x} e^{(\rho+\pi)\theta} e^{-IQ_i^\alpha(p, \rho, \bar{\theta}) Q_\alpha^i(k, \pi, \bar{\theta})}, \tag{5.53}$$

where  $Q_\alpha^i(p, \rho, \bar{\theta}) = (-\rho_\alpha^i + \bar{\theta}^{i\dot{\alpha}} \sigma_{\alpha\dot{\alpha}}^m p_m)$ . As a result, the equation (5.50) reads

$$\begin{aligned}
\Gamma_n &= \frac{(-1)^{n+1}}{n} \int d^4 \bar{\theta}_1 d^4 \bar{\theta}_n \frac{d^4 p_1 \dots d^4 p_n}{(2\pi)^{4n}} d^4 \rho_1 \dots d^4 \rho_n d^4 k_n d^4 \pi_n d^n U X_n[U] \\
& \delta^4(\bar{\theta}_1 - \bar{\theta}_n) \delta^4(\sum p_i) \delta^4(\sum \rho_i) \frac{\exp[\sum_{i,j=2(i < j)}^n - IQ(\rho_i) Q(\rho_j)]}{k_n^2 (k_n - p_1)^2 \dots (k_n - \sum_{i=1}^{n-1} p_i)^2} \\
& (D_{n-1}^+)^4 [(D_{n-2}^+)^4 [\dots [(D_2^+)^4 [(D_1^+)^4 (D_n^+)^4 \delta^4(\bar{\theta}_n - \bar{\theta}_1)] \\
& V^{++}(p_1, \rho_1, \bar{\theta}_1, u_1)] V^{++}(p_2, \rho_2, \bar{\theta}_1, u_2)] \dots \\
& V^{++}(p_{n-2}, \rho_{n-2}, \bar{\theta}_1, u_{n-2})] V^{++}(p_{n-1}, \rho_{n-1}, \bar{\theta}_1, u_{n-1}) V^{++}(p_n, \rho_n, \bar{\theta}_n, u_n). \tag{5.54}
\end{aligned}$$

Note that in this representation the derivatives  $\bar{D}_\alpha^+$  differentiate only  $\bar{\theta}$  variables while  $D_\alpha^+(\pi, u)$  is nothing but a multiplication operator on  $u_i^+ \pi_\alpha^i$ . Hence, we can apply the following identities

$$\frac{1}{16} \delta^4(\bar{\theta}_1 - \bar{\theta}_n) (\bar{D}_1^+)^2 (\bar{D}_n^+)^2 \delta^4(\bar{\theta}_n - \bar{\theta}_1) = (u_1^+ u_n^+)^2, \tag{5.55}$$

$$\frac{1}{16} \int d^4 \pi (D^+(\pi, u_{n-1}))^2 (D^+(\pi, u_n))^2 = (u_{n-1}^+ u_n^+)^2 \tag{5.56}$$

to simplify (5.54). After these manipulations we are left with  $n - 2$  differential operators in (5.54) which give at most the momentum  $(k^2)^{n-2}$  on condition that these derivatives do not hit the external lines. Clearly, this corresponds to the logarithmic divergence of the momentum

integral over  $d^4k$ . The terms with derivatives on the external lines  $V^{++}$  are always finite since they have the power of momentum less than  $n-2$ . Here we are interested only in the divergent contributions to the effective action and consider therefore only the terms in (5.54) without derivatives on  $V^{++}$  superfield. As a result, the divergent part of the effective action (5.54) is given by

$$\begin{aligned}\Gamma_{n,div} &= \frac{(-1)^{n+1}}{n} \frac{1}{16^n} \int \frac{d^4p_1}{(2\pi)^4} du_1 d^4\rho_1 \dots \frac{d^4p_n}{(2\pi)^4} du_n d^4\rho_n d^4\bar{\theta}_1 d^4\bar{\theta}_n d^4\pi d^4k \\ &\times V^{++}(p_1, \rho_1, \bar{\theta}_1, u_1) \dots V^{++}(p_n, \rho_n, \bar{\theta}_n, u_n) \exp[-I \sum_{i,j=2; i < j}^n Q(\rho_i) Q(\rho_j)] \\ &\times \delta^4(\bar{\theta}_1 - \bar{\theta}_n) (D_{n-1}^+)^2 (\bar{D}_{n-1}^+)^2 \dots (D_2^+)^2 (\bar{D}_2^+)^2 (D_1^+)^2 (\bar{D}_1^+)^2 (D_n^+)^2 (\bar{D}_n^+)^2 \delta^4(\bar{\theta}_1 - \bar{\theta}_n) \\ &\times \frac{\delta^4(\sum p_i) \delta^4(\sum \rho_i)}{k^2 (k - p_2)^2 \dots (k - \sum_{l=2}^n p_l)^2} \frac{1}{(u_1^+ u_2^+)^3 (u_2^+ u_3^+)^3 \dots (u_n^+ u_1^+)^3}.\end{aligned}\quad (5.57)$$

Note that the factor  $e^{-I \sum Q Q}$  in (5.57) allows us to restore the  $\star$ -product of gauge superfields  $V^{++}(u_1) \star V^{++}(u_2) \star \dots \star V^{++}(u_n)$  after the inverse Fourier transform.

Now we simplify the chain of covariant derivatives in (5.57). Consider, e.g., the block  $(D_2^+)^2 (\bar{D}_2^+)^2 (D_1^+)^2 (\bar{D}_1^+)^2$ . Using the identity  $D_{1\alpha}^+ = (u_1^+ u_2^-) D_{2\alpha}^+ - (u_1^+ u_2^+) D_{2\alpha}^-$  and anticommutation relations for the derivatives,  $\{D_\alpha^+, \bar{D}_{\dot{\alpha}}^-\} = 2k_{\alpha\dot{\alpha}}$ , this expression simplifies to

$$(D_2^+)^2 (\bar{D}_2^+)^2 (D_1^+)^2 (\bar{D}_1^+)^2 \longrightarrow 16 (u_1^+ u_2^+)^2 k^2 (D_2^+)^2 (\bar{D}_1^+)^2. \quad (5.58)$$

Applying the relation (5.58)  $n-2$  times, we rewrite the chain of derivatives in (5.57) as

$$\begin{aligned}&\delta^4(\bar{\theta}_1 - \bar{\theta}_n) (D_{n-1}^+)^2 (\bar{D}_{n-1}^+)^2 \dots (D_2^+)^2 (\bar{D}_2^+)^2 (D_1^+)^2 (\bar{D}_1^+)^2 (D_n^+)^2 (\bar{D}_n^+)^2 \delta^4(\bar{\theta}_1 - \bar{\theta}_n) \\ &= 16^{n-2} (k^2)^{n-2} (u_1^+ u_2^+)^2 \dots (u_{n-2}^+ u_{n-1}^+)^2 (D_n^+)^2 (\bar{D}_n^+)^2 \delta^4(\bar{\theta}_1 - \bar{\theta}_n) (\bar{D}_1^+)^2 (\bar{D}_n^+)^2 \delta^4(\bar{\theta}_1 - \bar{\theta}_n).\end{aligned}\quad (5.59)$$

Finally, integrating over  $d^4\bar{\theta}_n d^4\pi_n$  in (5.57) and using the identities (5.55), (5.56), we obtain

$$\Gamma_{n,div} = \frac{1}{16\pi^2\varepsilon} \frac{(-1)^n}{n} \int d^{12}z du_1 \dots du_n \frac{V^{++}(u_1) \star V^{++}(u_2) \star \dots \star V^{++}(u_n)}{(u_1^+ u_2^+) (u_2^+ u_3^+) \dots (u_n^+ u_1^+)}. \quad (5.60)$$

The factor  $\frac{1}{16\pi^2\varepsilon}$  appears here due to the dimensional regularization of logarithmically divergent momentum integral.

The full divergence in the model of charged hypermultiplet is obtained now by summarizing the divergent parts of  $n$ -point functions given by (5.60),

$$\Gamma_{div} = \frac{1}{16\pi^2\varepsilon} \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \int d^{12}z du_1 \dots du_n \frac{V^{++}(u_1) \star V^{++}(u_2) \star \dots \star V^{++}(u_n)}{(u_1^+ u_2^+) (u_2^+ u_3^+) \dots (u_n^+ u_1^+)}. \quad (5.61)$$

As a result, we conclude that the divergent part of the effective action in the charged hypermultiplet model coincides, up to a divergent factor, with the classical action (3.11) in the model of gauge superfield. This proves the renormalizability of the  $\mathcal{N}=(1,0)$  model of gauge superfield interacting with the hypermultiplet.



## 5.4 Seiberg-Witten transform and renormalizability

In this section we have proven the renormalizability of abelian models of gauge multiplet and hypermultiplets by direct computations of divergent contribution to the effective actions of these models. Recall that there is a change of classical fields in these actions (4.54), (4.87), (4.100) which not only brings the supersymmetry and gauge transformations to the undeformed form but also essentially simplifies the structures of the classical action. We refer to these transformations as the Seiberg-Witten maps. The Seiberg-Witten maps prove also very useful for the proof of renormalizability and finiteness of neutral hypermultiplet and gauge multiplet models. The quantum computations in terms of the transformed fields explicitly demonstrate the exact cancellations of divergent terms in the corresponding effective actions.

The Seiberg-Witten map in the abelian  $\mathcal{N}=(1,0)$  supergauge model (4.38) was derived in Sect. 4.2. Here we use these transformations in the form (4.59),

$$\begin{aligned}
\phi &\rightarrow \hat{\phi} = \phi + \frac{4I}{1+4I\bar{\phi}}[A_m A_m + 4I^2(\partial_m \bar{\phi})^2], \\
A_m &\rightarrow a_m = \frac{A_m}{1+4I\bar{\phi}}, \quad \bar{\Psi}_{\dot{\alpha}}^k \rightarrow \bar{\psi}_{\dot{\alpha}}^k = \frac{\bar{\Psi}_{\dot{\alpha}}^k}{1+4I\bar{\phi}}, \\
\Psi_{\alpha}^k &\rightarrow \hat{\psi}_{\alpha}^k = \Psi_{\alpha}^k + \frac{4I A_{\alpha\dot{\alpha}} \bar{\Psi}^{\dot{\alpha}k}}{1+4I\bar{\phi}}, \\
\mathcal{D}^{kl} &\rightarrow \hat{d}^{kl} = \frac{1}{1+4I\bar{\phi}} \left[ \mathcal{D}^{kl} + \frac{8I \bar{\Psi}_{\dot{\alpha}}^k \bar{\Psi}^{\dot{\alpha}l}}{1+4I\bar{\phi}} \right].
\end{aligned} \tag{5.62}$$

The action (4.38) in terms of these new fields is rewritten as

$$\begin{aligned}
S_{SYM} &= \int d^4x \left( -\frac{1}{2} \hat{\phi} \square \bar{\phi} - i \hat{\psi}_{\dot{\alpha}}^k \partial_{\alpha\dot{\alpha}} \bar{\psi}^{\dot{\alpha}k} + \frac{1}{4} \hat{d}^{kl} \hat{d}_{kl} \right) \\
&\quad + \frac{1}{4} \int d^4x (1+4I\bar{\phi})^2 (f_{mn} f_{mn} + \frac{1}{2} \varepsilon_{mnrs} f_{mn} f_{rs}),
\end{aligned} \tag{5.63}$$

where  $f_{mn} = \partial_m a_n - \partial_n a_m$ . The form (5.63) is more preferable for further quantum computations as compared to the one given by (4.57), since the scalar  $\hat{\phi}$ , as well as the spinor and auxiliary fields, are free in (5.63). The only interaction term is present in the second line of (5.63). It is an interaction between the vector field strength and the scalar  $\bar{\phi}$ .

The action (5.63) is invariant under the abelian gauge transformation,

$$\delta a_m = \partial_m \lambda, \tag{5.64}$$

$\lambda$  being the gauge parameter. We use here the Lorentz gauge

$$\partial_m a_m = 0, \tag{5.65}$$

since the transformation (5.64) has the same form as in the classical electrodynamics. Further we follow the Faddeev-Popov procedure of constructing the functional integral. Let us introduce the gauge-fixing function

$$\chi = \partial_m a_m, \tag{5.66}$$

which transforms under (5.64) as  $\delta\chi = \square\lambda$ . Obviously, the ghost fields do not interact with any other ones and so they completely decouple. The ghost action is given again by (5.7). The generating functional for Green's functions is now given by <sup>7</sup>

$$Z[J] = \int \mathcal{D}(\hat{\varphi}, \bar{\phi}, \hat{\psi}, \bar{\psi}, a_m, b, c) \delta(\chi - \partial_m a_m) e^{-\frac{1}{2}(S_{SYM} + S_{FP} + S_J)}, \quad (5.67)$$

where

$$S_J = \int d^4x [\hat{\varphi} J_{\hat{\varphi}} + \bar{\phi} J_{\bar{\phi}} + \hat{\psi}_\alpha^i (J_{\hat{\psi}})_i^\alpha + \bar{\psi}_{i\dot{\alpha}} (J_{\bar{\psi}})^{i\dot{\alpha}} + a_m (J_a)_m]. \quad (5.68)$$

To represent the delta-function in the Gaussian form, we average the equation (5.67) with the functional factor (5.10). As a result, we obtain the gauge-fixing action in the form

$$S_{gf} = \frac{\alpha}{2} \int d^4x (1 + 4I\bar{\phi})^2 \partial_m a_m \partial_n a_n. \quad (5.69)$$

For simplicity we choose the gauge -fixing parameter  $\alpha$  to be equal to unity,  $\alpha = 1$ . Now, the generating functional (5.67) reads

$$Z[J] = \int \mathcal{D}(\hat{\varphi}, \bar{\phi}, \hat{\psi}, \bar{\psi}, a_m, b, c) e^{-\frac{1}{2}(S_{\text{tot}} + S_{FP} + S_J)}, \quad (5.70)$$

where

$$S_{\text{tot}} = S_{SYM} + S_{gf} = \int d^4x \left( -\frac{1}{2} \hat{\varphi} \square \bar{\phi} - i \hat{\psi}_k^\alpha \partial_{\alpha\dot{\alpha}} \bar{\psi}^{\dot{\alpha}k} + \frac{1}{4} \hat{d}^{kl} \hat{d}_{kl} \right) + S_a, \quad (5.71)$$

and

$$S_a = \frac{1}{2} \int d^4x (1 + 4I\bar{\phi})^2 (\partial_m a_m \partial_n a_n + \partial_m a_n \partial_m a_n - \partial_m a_n \partial_n a_m + \varepsilon_{mnrs} \partial_m a_n \partial_r a_s). \quad (5.72)$$

It is evident that the scalar and spinor fields, as well as the ghosts, do not contribute to the effective action. The only contribution comes from the part (5.72), namely

$$\begin{aligned} \Gamma^{SYM} &= \frac{1}{2} \text{Tr} \ln \frac{\delta^2 S_a}{\delta a_p(x) \delta a_q(x')} \\ &= \frac{1}{2} \text{Tr} \ln [\delta_{pq} \square \delta^4(x - x') + 2\delta_{pq} G_m \partial_m \delta^4(x - x') \\ &\quad + 4G_{[p} \partial_{q]} \delta^4(x - x') - 2\varepsilon_{pqmn} G_m \partial_n \delta^4(x - x')]. \end{aligned} \quad (5.73)$$

The field  $G_m(x)$  was defined in (5.5). The expression (5.73) is the starting point for perturbative calculations of one-loop effective action in the  $\mathcal{N}=(1,0)$  non-anticommutative SYM model. Note that it resembles the first line of (5.18), except for the term  $2\delta_{pq} G_m \partial_m \delta^4(x - x')$ . Therefore the further computations are very similar to the ones in Sect. 2. As usual, only two-, three- and four-point diagrams are divergent. The two-point function is given by

$$\begin{aligned} \Gamma_2^{SYM} &= - \int d^4x_1 d^4x_2 [\delta_{pq} G_m(x_1) \partial_m \frac{1}{\square} \delta^4(x_1 - x_2) + 2G_{[p}(x_1) \partial_{q]} \frac{1}{\square} \delta^4(x_1 - x_2) \\ &\quad + \varepsilon_{qpmn} G_m(x_1) \partial_n \frac{1}{\square} \delta^4(x_1 - x_2)] [\delta_{qp} G_n(x_2) \partial_n \frac{1}{\square} \delta^4(x_2 - x_1) \\ &\quad + 2G_{[q}(x_2) \partial_{p]} \frac{1}{\square} \delta^4(x_2 - x_1) + \varepsilon_{pqrs} G_r(x_2) \partial_s \frac{1}{\square} \delta^4(x_2 - x_1)]. \end{aligned} \quad (5.74)$$

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<sup>7</sup>Note that the Jacobian of the change of functional variables (5.62) is unity since this redefinition of fields is local.

To proceed, we pass to momentum space and compute the divergent momentum integrals using the standard methods of quantum field theory. As a result, we find that the two-point function (5.74) has no divergent contributions,

$$\Gamma_{2,div}^{SYM} = 0. \quad (5.75)$$

The absence of divergences here is owing to the term  $2\delta_{pq}G_m\partial_m\delta^4(x-x')$  in (5.73) and (5.74). It gives the contribution which exactly cancels the expression (5.23) obtained by similar calculations without this term.

The three- and four-point functions are defined by the following formal expressions

$$\Gamma_3^{SYM} = \frac{4}{3}\text{Tr}[(\delta_{pq}G_m(x)\partial_m + 2G_{[p}(x)\partial_{q]} - \varepsilon_{pqmn}G_m(x)\partial_n)\frac{1}{\square}\delta^4(x-x')]^3, \quad (5.76)$$

$$\Gamma_4^{SYM} = -2\text{Tr}[(\delta_{pq}G_m(x)\partial_m + 2G_{[p}(x)\partial_{q]} - \varepsilon_{pqmn}G_m(x)\partial_n)\frac{1}{\square}\delta^4(x-x')]^4. \quad (5.77)$$

The further computations are very similar to those in Sect. 5.1, but with taking into account the term  $2\delta_{pq}G_m\partial_m\delta^4(x-x')$ . After carefully tracking the coefficients during the computations, we find that the three- and four-point functions also have no divergences,

$$\Gamma_{3,div}^{SYM} = 0, \quad \Gamma_{4,div}^{SYM} = 0. \quad (5.78)$$

As a result, we conclude that the abelian  $\mathcal{N}=(1,0)$  non-anticommutative gauge model (5.63) is completely finite,

$$\Gamma_{div}^{SYM} = 0, \quad (5.79)$$

without the necessity to perform any field redefinition such as (5.27).

One more important comment to be added is as follows. The abelian  $\mathcal{N}=(1,0)$  non-anticommutative gauge model is described by the classical actions (4.38) or (5.63) which are related to each other by the Seiberg-Witten map (5.62). It is obvious that the Jacobian of such a change of functional variables (5.62) is unity (in the sense of dimensional regularization). Therefore the effective actions in these two models should also be related by the Seiberg-Witten map. As for the divergent part, we observe that it is trivial for both models (4.38) and (5.63), since it can be removed by the shift (5.27) of the scalar field  $\phi$ . Note that this explains the appearance of only two out of three possible divergent terms (5.14). Indeed, if the third term proportional to  $I^4 \int d^4x f_3(I\bar{\phi})(\partial_m\bar{\phi}\partial_m\bar{\phi})^2$  appeared in the divergent part of the effective action, it could not be removed by any shift of the scalar field  $\phi$ , that would mean the presence of a nontrivial divergence in the model. However, we have seen in this Section that the effective action in  $\mathcal{N}=(1,0)$  non-anticommutative gauge theory is finite.

Let us also consider the general model of an abelian  $\mathcal{N}=(1,0)$  non-anticommutative gauge superfield interacting with a neutral hypermultiplet. It is described by the sum of the classical actions (4.38) and (4.81). Upon performing the Seiberg-Witten map (4.89), this action turns into (4.90). We see that the non-trivial interaction terms in (4.92) are the ones given by

$$\int d^4x \frac{1}{4}(1 + 4I\bar{\phi})^2(f_{mn}f_{mn} + \frac{1}{2}\varepsilon_{mnrs}f_{mn}f_{rs}). \quad (5.80)$$

This expression just coincides with the one present in the gauge theory action (5.63). Thus the quantum computations tell us once again that the general abelian  $\mathcal{N}=(1,0)$  non-anticommutative model is finite

$$\Gamma_{div} = 0. \quad (5.81)$$

This result agrees with the one of Sect. 5.2, modulo some divergent redefinition (5.42) of the scalar field  $\phi$ .

To summarize, the use of the Seiberg-Witten map in the models under consideration makes it possible to avoid the divergent expressions in the effective action from the very beginning. Otherwise, such expressions appear but they are removable by some divergent redefinition of the scalar field  $\phi$ .

## 6 Holomorphic potential in the non-anticommutative abelian charged hypermultiplet model

The previous section was devoted to the calculation of divergent parts of the effective actions in the models of gauge superfield and hypermultiplets. In this section we study the structure of finite parts of these effective actions. It is clear that the chiral singlet deformation modifies the effective action of the original undeformed theory in some way. Since we restrict our consideration only to the abelian case, it makes sense to study only the issue of non-anticommutative corrections to the effective action in the charged hypermultiplet model. Indeed, in the limit  $I \rightarrow 0$  the undeformed abelian models of neutral hypermultiplet and gauge superfield become free and exhibit no any quantum dynamics, while the charged hypermultiplet model turns into the  $\mathcal{N}=(1, 1)$  supersymmetric electrodynamics which is non-trivial at the quantum level.

It is well known [37, 39] that the low-energy effective action in the undeformed charged hypermultiplet model has the following structure in the sector of gauge superfields

$$\Gamma = \int d^4x d^4\theta \mathcal{F}(W) + \int d^4x d^4\bar{\theta} \bar{\mathcal{F}}(\bar{W}) + \int d^4x d^8\theta \mathcal{H}(W, \bar{W}), \quad (6.1)$$

where  $\mathcal{F}$  is a holomorphic potential,  $\bar{\mathcal{F}}$  is an antiholomorphic potential and  $\mathcal{H}$  is a non-holomorphic potential. The superfield strengths  $W, \bar{W}$  are expressed through the prepotential  $V^{--}$  as in (3.4). In the abelian case these superfields obey the (anti)chirality conditions,

$$D_\alpha^\pm \bar{W} = 0, \quad \bar{D}_{\dot{\alpha}}^\pm W = 0. \quad (6.2)$$

The holomorphic and antiholomorphic parts of the effective action (6.1) result in the following effective equations of motion

$$(D^+)^2 \mathcal{F}'(W) + (\bar{D}^+)^2 \bar{\mathcal{F}}'(\bar{W}) = 0. \quad (6.3)$$

By now, the perturbative contributions to the effective action in the undeformed charged hypermultiplet model have been thoroughly studied (see, e.g., [37]-[42]). In particular, the holomorphic potential in this model is given by the following simple formula

$$\mathcal{F}(W) = -\frac{1}{32\pi^2} W^2 \ln \frac{W}{\mu}, \quad (6.4)$$

where  $\mu$  is some constant of mass dimension +1.

The superfield strengths  $W, \bar{W}$  have the scalar fields  $\phi, \bar{\phi}$  and the Maxwell field strength  $F_{mn} = \partial_m A_n - \partial_n A_m$  as their bosonic component fields,

$$W = \phi + (\theta^+ \sigma_{mn} \theta^-) F_{mn} + \dots, \quad \bar{W} = \bar{\phi} + (\bar{\theta}^+ \tilde{\sigma}_{mn} \bar{\theta}^-) F_{mn} + \dots \quad (6.5)$$

Here we do not consider the dependence of these superfields on the spinors  $\Psi_\alpha^i$ ,  $\bar{\Psi}_{i\dot{\alpha}}$  and auxiliary fields  $\mathcal{D}^{kl}$ . Substituting the field strength (6.5) into (6.4), one readily obtains the component structure of the holomorphic effective action in the bosonic sector,

$$\Gamma_{hol} = \int d^4x d^4\theta \mathcal{F}(W) = -\frac{1}{32\pi^2} \int d^4x (F_{mn}F_{mn} + F_{mn}\tilde{F}_{mn}) \left( \ln \frac{\phi}{\mu} + \frac{3}{2} \right) + \dots \quad (6.6)$$

Here  $\tilde{F}_{mn} = \frac{1}{2}\varepsilon_{mnrs}F_{rs}$  and dots stand for the terms with derivatives of fields and spinor fields. We stress that the expression (6.6) corresponds to the bosonic terms in the effective action which are leading in the following approximation

$$\begin{aligned} \phi = const, \quad \bar{\phi} = const, \quad F_{mn} = const, \\ \Psi_\alpha^i = \bar{\Psi}_{i\dot{\alpha}} = \mathcal{D}^{kl} = 0. \end{aligned} \quad (6.7)$$

Note that the constant 3/2 in (6.6) can be removed by a shift of the parameter  $\mu$ , however it will be important when we will consider the non-anticommutative deformation of (anti)holomorphic effective action. The antiholomorphic part of the effective action can be obtained by the complex conjugation of the action (6.6).

## 6.1 General structure of the effective action

Let us discuss the general structure of the effective action in the charged hypermultiplet model. Since the classical action (3.14) is a simple  $\star$ -product generalization of the corresponding classical action of undeformed theory, one can assume that the chiral part of the effective potentials in (6.1) is also given by the  $\star$ -deformation of the holomorphic potential,

$$\mathcal{F}(W) \longrightarrow \mathcal{F}_\star(W). \quad (6.8)$$

However, the antiholomorphic contributions to the effective action cannot be accounted by such a naive considerations. As was shown in Sect. 4.1, one cannot construct any action in the antichiral superspace having the form  $\bar{W}_\star^n$ . We will show that the corresponding contributions to the effective action are naturally given by the full superspace integrals.

For the further consideration it will be more convenient to study the variation of effective action  $\delta\Gamma$ , rather than  $\Gamma$  itself. In particular, given the holomorphic effective action

$$\Gamma_{hol} = \int d^4x d^4\theta \mathcal{F}_\star(W), \quad (6.9)$$

one can write its variation either in the analytic superspace,

$$\delta\Gamma_{hol} = \int d\zeta du \delta V^{++} \star \left[ -\frac{1}{4} D^{+\alpha} D_\alpha^+ \mathcal{F}'_\star(W) \right], \quad (6.10)$$

or in full superspace,

$$\delta\Gamma_{hol} = \int d^{12}z du \delta V^{++} \star V^{--} \star \frac{1}{W} \star \mathcal{F}'_\star(W). \quad (6.11)$$

To derive the expressions (6.10), (6.11) one should follow the same steps as in [42] for the non-abelian  $\mathcal{N}=2$  supersymmetric gauge model without deformations.

We assume that the antiholomorphic contributions to the effective action can be accounted by the following variation

$$\delta\Gamma_{antihol} = \int d^{12}z du \delta V^{++} \star V^{--} \star \frac{1}{\bar{W}} \star \bar{\mathcal{F}}'_\star(\bar{W}), \quad (6.12)$$

which is written in full  $\mathcal{N}=(1,1)$  superspace rather than in the antichiral one. In particular, it reproduces the correct effective equations of motion of the form (6.3). Clearly, the full superspace action  $\delta\Gamma_{antihol}$  can always be reduced to the antichiral superspace by integrating over  $d^4\theta$ , but the result cannot be written as  $\int d^4x d^4\bar{\theta} \bar{\mathcal{F}}_\star(\bar{W})$ .

## 6.2 One-loop effective action

In this subsection we explicitly calculate the leading contributions to the charged hypermultiplet effective action in harmonic superspace. Consider the full propagator  $G^{(1,1)}(1|2)$  of the charged hypermultiplet defined as a solution of the equation

$$\nabla^{++} \star G^{(1,1)}(1|2) = \delta_A^{(3,1)}(1|2). \quad (6.13)$$

In contrast to the free propagator  $G_0^{(1,1)}(1|2)$  given by (5.44),  $G^{(1,1)}(1|2)$  describes the dynamics of the charged hypermultiplet interacting with the background gauge superfield  $V^{++}$ . It is straightforward to check that the solution of (6.13) can be written as

$$G^{(1,1)}(1|2) = -\frac{1}{\hat{\square}_\star} \star (D_1^+)^4 (D_2^+)^4 \left\{ e_\star^{\Omega(1)} \star e_\star^{-\Omega(2)} \star \frac{\delta^{12}(z_1 - z_2)}{(u_1^+ u_2^+)^3} \right\}, \quad (6.14)$$

where  $\hat{\square}_\star$  is the covariant box operator,

$$\hat{\square}_\star = -\frac{1}{2}(D^+)^4 \nabla^{--} \star \nabla^{--}, \quad (6.15)$$

and  $\Omega(z, u)$  is a “bridge” superfield in the full  $\mathcal{N}=(1,1)$  superspace defined by the relations

$$\nabla^{++} = e_\star^\Omega \star D^{++} e_\star^{-\Omega}, \quad \nabla^{--} = e_\star^\Omega \star D^{--} e_\star^{-\Omega}. \quad (6.16)$$

The bridge superfield was originally introduced in [34] for the undeformed  $\mathcal{N}=2$  supergauge theory as an operator relating  $\mathcal{N}=2$  superfields in the so-called  $\lambda$ - and  $\tau$ -frames. Using the superfield  $\Omega(z, u)$  one can write the relation (3.3) between the prepotentials  $V^{--}$  and  $V^{++}$  in the following simple forms,

$$V^{--}(z, u) = \int du' \frac{e_\star^{\Omega(z, u)} \star e_\star^{-\Omega(z, u')}\star V^{++}(z, u')}{(u^+ u'^+)^2} = \int du' \frac{V^{++}(z, u') \star e_\star^{\Omega(z, u')}\star e_\star^{-\Omega(z, u)}}{(u^+ u'^+)^2}. \quad (6.17)$$

The relations (6.17) can be directly checked using (6.16) and the properties of harmonic distributions.

Note that the operator  $\hat{\square}_\star$  takes any analytic superfield into an analytic one. This operator, while acting on analytic superfields, can be represented in the following form

$$\begin{aligned} \hat{\square}_\star &= \nabla^m \star \nabla_m - \frac{1}{2}(\nabla^{+\alpha} \star W) \star \nabla_\alpha^- - \frac{1}{2}(\bar{\nabla}_\alpha^+ \star \bar{W}) \star \bar{\nabla}^{-\dot{\alpha}} + \frac{1}{4}(\nabla^{+\alpha} \star \nabla_\alpha^+ \star W) \star \nabla^{--} \\ &\quad - \frac{1}{8}[\nabla^{+\alpha} \star \nabla_\alpha^-] \star W - \frac{1}{2}\{W \star \bar{W}\}, \end{aligned} \quad (6.18)$$

where  $\nabla_\alpha^\pm = D_\alpha^\pm + V_\alpha^\pm$ ,  $\bar{\nabla}_{\dot{\alpha}}^\pm = \bar{D}_{\dot{\alpha}}^\pm + \bar{V}_{\dot{\alpha}}^\pm$  are covariant spinor derivatives. The expression (6.18) has the same form as in the undeformed non-abelian gauge theory [40], with the  $\star$ -product playing the role of the matrix commutator. This result is not surprising since (6.18) is derived using only the algebra of covariant derivatives which has the same form as in the undeformed case.

Clearly, the charged hypermultiplet effective action is one-loop exact since the classical action (3.14) is quadratic in the hypermultiplet superfields. It can also be expressed through the propagator  $G^{(1,1)}(1|2)$ ,

$$\Gamma = \text{Tr} \ln \frac{\delta^2 S}{\delta \tilde{q}^+(1) \delta q^+(2)} = \text{Tr} \ln(\nabla^{++} \star) = -\text{Tr} \ln G^{(1,1)}(1|2). \quad (6.19)$$

The variation of this effective action reads

$$\delta\Gamma = \text{Tr}[\delta V^{++} \star G^{(1,1)}] = \int d\zeta du \delta V^{++}(1) \star G^{(1,1)}(1|2)|_{(1)=(2)}. \quad (6.20)$$

Using the definition (6.13), one can derive the following relation between the free and full hypermultiplet propagators,

$$G^{(1,1)}(1|3) = G_0^{(1,1)}(1|3) - \int d\zeta_2 du_2 G_0^{(1,1)}(1|2) \star V^{++}(2) \star G^{(1,1)}(2|3). \quad (6.21)$$

Substituting (6.21) into the variation (6.20), we find

$$\delta\Gamma = - \int d\zeta_1 du_1 d\zeta_2 du_2 \delta V^{++}(1) \star G_0^{(1,1)}(1|2) \star V^{++}(2) \star G^{(1,1)}(2|1). \quad (6.22)$$

Taking into account the explicit forms of the propagators (5.44), (6.14), we rewrite (6.22) as follows

$$\begin{aligned} \delta\Gamma = & - \int d\zeta_1 d\zeta_2 du_1 du_2 \delta V^{++}(1) \star \frac{1}{\square} (D_1^+)^4 (D_2^+)^4 \frac{\delta^{12}(z_1 - z_2)}{(u_1^+ u_2^+)^3} \\ & \times V^{++}(2) \star \frac{1}{\hat{\square}_{\star(2)}} \star (D_1^+)^4 (D_2^+)^4 \left\{ e_\star^{\Omega(2)} \star e_\star^{-\Omega(1)} \frac{\delta^{12}(z_2 - z_1)}{(u_2^+ u_1^+)^3} \right\}. \end{aligned} \quad (6.23)$$

Now we take off the spinor derivatives from the first delta-function to restore full  $\mathcal{N}=(1,1)$  superspace measure with the help of (5.48),

$$\begin{aligned} \delta\Gamma = & - \int d^{12}z_1 d^{12}z_2 du_1 du_2 \delta V^{++}(1) \star \frac{1}{\square} \frac{\delta^{12}(z_1 - z_2)}{(u_1^+ u_2^+)^3} \\ & \times V^{++}(2) \star \frac{1}{\hat{\square}_{\star(2)}} \star (D_1^+)^4 (D_2^+)^4 \left\{ e_\star^{\Omega(2)} \star e_\star^{-\Omega(1)} \star \frac{\delta^{12}(z_2 - z_1)}{(u_2^+ u_1^+)^3} \right\}. \end{aligned} \quad (6.24)$$

We did not impose any restrictions on the background gauge superfields so far, therefore (6.24) is the exact representation for the hypermultiplet effective action. It should be considered as a starting point for further calculations of different contributions to the effective action.

### 6.3 Divergent part of the effective action

The effective action, as a functional of superfield strengths, can be expanded in a series with respect to these superfields and their covariant derivatives. This series can be obtained from the representation (6.24) for the effective action as a result of the decomposition of the operator  $1/\hat{\square}_\star$  in this expression.

Let us omit all superfields in the operator (6.18),

$$\frac{1}{\hat{\square}_\star} \approx \frac{1}{\square}. \quad (6.25)$$

Such an approximation, being applied to (6.24), corresponds exactly to the divergent part of the effective action since the other terms in the covariant box operator produce higher powers of momenta in the denominator which lead to the finite contributions. Under the condition (6.25), the variation of the effective action (6.24) is essentially simplified

$$\begin{aligned} \delta\Gamma_{div} = & \int d^{12}z_1 d^{12}z_2 \frac{du_1 du_2}{(u_1^+ u_2^+)^6} \delta V^{++}(1) \star \frac{1}{\square} \delta^{12}(z_1 - z_2) \\ & \times V^{++}(2) \star \frac{1}{\square} (D_1^+)^4 (D_2^+)^4 \{ e_\star^{\Omega(2)} \star e_\star^{-\Omega(1)} \star \delta^{12}(z_2 - z_1) \}. \end{aligned} \quad (6.26)$$

Next, we apply the identity

$$\delta^8(\theta_1 - \theta_2) (D_1^+)^4 (D_2^+)^4 \delta^{12}(z_1 - z_2) = (u_1^+ u_2^+)^4 \delta^{12}(z_1 - z_2) \quad (6.27)$$

to shrink the integration over the Grassmann variables to a point. As a result, after regularization of the divergent momentum integral, (6.26) becomes

$$\delta\Gamma_{div} = \frac{1}{16\pi^2\varepsilon} \int d^{12}z du_1 \delta V^{++}(z, u_1) \star \int du_2 \frac{V^{++}(z, u_2) \star e_\star^{\Omega(z, u_2)} \star e_\star^{-\Omega(z, u_1)}}{(u_1^+ u_2^+)^2}. \quad (6.28)$$

Here  $\varepsilon$  is a parameter of dimensional regularization. Applying now (6.17), we obtain the following expression for the variation of the effective action

$$\delta\Gamma_{div} = \frac{1}{16\pi^2\varepsilon} \int d^{12}z du \delta V^{++} \star V^{--}. \quad (6.29)$$

The variation (6.29) can be easily integrated with the help of (6.9), (6.11),

$$\Gamma_{div} = \frac{1}{32\pi^2\varepsilon} \int d^4x d^4\theta W^2. \quad (6.30)$$

As a result, we see that the divergent part of the effective action is proportional to the classical action of the  $\mathcal{N}=(1, 0)$  supersymmetric gauge theory. This result has already been obtained in Sect. 5.3 by a different method.



## 6.4 Holomorphic and non-holomorphic contributions

In this subsection we will study the finite contributions to the effective action in the charged hypermultiplet model. The leading terms in the low-energy effective action are composed of the superfield strengths without derivatives. Such an approximation is effectively accounted by considering the background superfield strengths obeying the following constraints

$$\nabla^{\pm\alpha} \star W = 0, \quad \bar{\nabla}_{\dot{\alpha}}^{\pm} \star \bar{W} = 0. \quad (6.31)$$

Under the constraints (6.31) all superfields with derivatives in the operator  $\hat{\square}_{\star}$  given by (6.24) can be neglected<sup>8</sup>,

$$\frac{1}{\hat{\square}_{\star}} \approx \frac{1}{\square - \frac{1}{2}\{W \star \bar{W}\}}. \quad (6.32)$$

As a result, the variation of the effective action is given by

$$\begin{aligned} \delta\Gamma &= \int d^{12}z_1 d^{12}z_2 \frac{du_1 du_2}{(u_1^+ u_2^+)^6} \delta V^{++}(1) \star \frac{1}{\square} \delta^{12}(z_1 - z_2) \\ &\quad \times V^{++}(2) \star \frac{1}{\square - \frac{1}{2}\{W \star \bar{W}\}} (D_1^+)^4 (D_2^+)^4 \{e_{\star}^{\Omega(2)} \star e_{\star}^{-\Omega(1)} \star \delta^{12}(z_2 - z_1)\}. \end{aligned} \quad (6.33)$$

Next, we apply the identity (6.27) and integrate over  $d^8\theta_2$  using the corresponding delta-function

$$\begin{aligned} \delta\Gamma &= \int d^{12}z_1 d^4x_2 \frac{du_1 du_2}{(u_1^+ u_2^+)^2} \delta V^{++}(x_1, \theta, u_1) \star V^{++}(x_2, \theta, u_2) \star e_{\star}^{\Omega(2)} \star e_{\star}^{-\Omega(1)} \\ &\quad \star \frac{1}{\square} \delta^4(x_1 - x_2) \frac{1}{\square - \frac{1}{2}\{W \star \bar{W}\}} \delta^4(x_2 - x_1). \end{aligned} \quad (6.34)$$

The bosonic delta-functions in (6.34) result in the following momentum integral

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2} \frac{1}{k^2 + \frac{1}{2}\{W \star \bar{W}\}} = -\frac{1}{16\pi^2} \ln_{\star} \left[ \frac{\{W \star \bar{W}\}}{2\mu^2} \right] + (\text{divergent terms}), \quad (6.35)$$

where  $\mu$  is an arbitrary constant of dimension +1. The function  $\ln_{\star}$  here is understood in a sense of the corresponding Taylor series with the  $\star$ -product of superfields, e.g.,  $\ln_{\star}(1 + X) = X - \frac{1}{2}X \star X + \frac{1}{3}X \star X \star X + \dots$ . Since the divergent part of the effective action was studied in the previous subsection, here we concentrate only on the finite part. Applying the identity (6.17), we conclude,

$$\delta\Gamma = -\frac{1}{16\pi^2} \int d^{12}z du \delta V^{++} \star V^{--} \star \ln_{\star} \frac{\{W \star \bar{W}\}}{2\mu^2}. \quad (6.36)$$

The expression (6.36) is responsible for all contributions to the effective action with the superfield strengths without derivatives.

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<sup>8</sup>In (6.32) we discard also the connections covariantizing the vector derivatives  $\partial_m$  which are present in the first term of the operator (6.18). These connections are always proportional to the derivatives of superfield strengths and therefore are not essential for studies of the holomorphic effective action.

Note that in the limit  $I \rightarrow 0$  the  $\star$ -product becomes the usual multiplication and (6.36) is given by

$$\delta\Gamma_{(I=0)} = -\frac{1}{16\pi^2} \int d^{12}z du \delta V^{++} V^{--} \left( \ln \frac{W}{\mu} + \ln \frac{\bar{W}}{\mu} \right). \quad (6.37)$$

The variation (6.37) corresponds precisely to the holomorphic and antiholomorphic parts of the effective action (6.1) with the holomorphic potential (6.4) of the undeformed theory. However, if  $I \neq 0$ , the logarithm in (6.36) cannot be written as a sum of two logarithms since there are mixed terms. Therefore in the non-anticommutative case the expression (6.36) is responsible for both holomorphic, antiholomorphic and non-holomorphic contributions to the effective action.

We have to extract the holomorphic and antiholomorphic parts from the effective action (6.36). For this purpose we apply the following identity for the logarithm in (6.36)

$$\ln_{\star} \frac{\{W^{\star}; \bar{W}\}}{2\mu^2} = \ln_{\star} \frac{W}{\mu} + \ln_{\star} \frac{\bar{W}}{\mu} + \frac{1}{12\mu^3} [W^{\star}; [W^{\star}; \bar{W}]] + \frac{1}{12\mu^3} [\bar{W}^{\star}; [\bar{W}^{\star}; W]] + \dots, \quad (6.38)$$

where dots stand for terms of fourth and higher orders in superfields which come with various commutators. Note that the expression (6.38) is valid without any restrictions on the superfields and is obtained only with the help of formal manipulations with  $\star$ -(anti)commutators of superfields. The terms with commutators in (6.38) correspond to the non-holomorphic contributions to the effective action since they involve both  $W$  and  $\bar{W}$ . Keeping only the first two terms in the r.h.s. of (6.38), we obtain the following expression for the variation (6.36):

$$\delta\Gamma = -\frac{1}{16\pi^2} \int d^{12}z du \delta V^{++} \star V^{--} \star \left[ \ln_{\star} \frac{W}{\mu} + \ln_{\star} \frac{\bar{W}}{\mu} \right] + \dots \quad (6.39)$$

Here dots stand for the non-holomorphic contributions. According to the equation (6.11), the holomorphic part of the variation (6.39) can be easily integrated,

$$\Gamma_{hol} = -\frac{1}{32\pi^2} \int d^4x d^4\theta W \star W \star \ln_{\star} \frac{W}{\mu}. \quad (6.40)$$

As a result, we proved that the holomorphic part of the effective action in the non-anticommutative charged hypermultiplet model is nothing but a  $\star$ -product generalization of a standard holomorphic potential (6.4).

Note that the terms with commutators in (6.38) can be eliminated by imposing the further constraints on the background gauge superfields. Consider, e.g., the following constraints

$$\partial_m \bar{W} = 0, \quad \frac{\partial}{\partial \theta_{\alpha}^i} \bar{W} = 0. \quad (6.41)$$

One of the consequences of (6.41) is the relation  $Q_{\alpha}^i \bar{W} = 0$  which reduces the  $\star$ -product of the superfield strengths to the usual product. Note that the constraints (6.41) are not covariant and could be too strong. However, they keep the dependence of the superfield  $\bar{W}$  on the  $\bar{\theta}^{i\dot{\alpha}}$  variables and are consistent with the approximation (6.7) in which we study the corrections to the holomorphic potential. Moreover, the constraints (6.41) do not violate the covariance of the effective action in the holomorphic sector.

The constraint (6.41) simplifies the antiholomorphic part of the effective action since it allows one to omit the  $\star$ -product,

$$\delta\Gamma_{antihol} = \frac{1}{16\pi^2} \int d^{12}z du \delta V^{++} V^{--} \ln \frac{\bar{W}}{\mu}. \quad (6.42)$$

The variation (6.42) reproduces the standard antiholomorphic potential,

$$\Gamma_{antihol} = -\frac{1}{32\pi^2} \int d^4x d^4\bar{\theta} \bar{W}^2 \ln \frac{\bar{W}}{\mu}. \quad (6.43)$$

Despite the absence of  $\star$ -product in (6.43), this expression implicitly depends on the parameter of chiral singlet deformation  $I$  through the superfield  $\bar{W}$  which involves this parameter by definition.

## 6.5 Component structure of the effective action

The leading contributions to the effective action in the undeformed hypermultiplet model are given by (6.6). Here we study the corrections to these terms due to the non-anticommutative deformations of supersymmetry. For this purpose we find the component structure of the actions (6.40), (6.42) in the bosonic sector in the approximation (6.7). Here we follow the same steps as in the Sect. 4.1 where the component structure of the classical action of  $\mathcal{N}=(1,0)$  supergauge theory was studied.

In the component expansion of the prepotential (4.4) we keep only the bosonic fields,

$$V_{WZ}^{++} = (\theta^+)^2 \bar{\phi} + (\bar{\theta}^+) \phi + 2(\theta^+ \sigma_m \bar{\theta}^+) A_m - 2i(\bar{\theta}^+)^2 (\theta^+ \theta^-) \partial_m A_m - (\bar{\theta}^+)^2 (\theta^- \sigma_{mn} \theta^+) F_{mn}. \quad (6.44)$$

Note that both the strength  $F_{mn}$  and gauge potential  $A_m$  enter the prepotential (6.44). Therefore the expression (6.44) depends on the spatial coordinates  $x^m$  through the potential  $A_m$ . Without loss of generality, we choose the vector potential to be linear in  $x^m$ ,  $A_m = \frac{1}{2} F_{nm} x^n$ ,  $F_{mn} = \text{const}$ . In particular,  $\partial_m A_n - \partial_n A_m = F_{mn}$ ,  $\partial_m A_m = 0$ .

Analogously to the expression (4.14), we look for the prepotential  $V^{--}$  in the form

$$\begin{aligned} V^{--} = & v^{--} + \bar{\theta}_\alpha^- v^{-\dot{\alpha}} + (\bar{\theta}^-)^2 A + (\bar{\theta}^+ \bar{\theta}^-) \varphi^{--} \\ & + (\bar{\theta}^+ \bar{\sigma}_{mn} \bar{\theta}^-) \varphi_{mn}^{--} + (\bar{\theta}^-)^2 \bar{\theta}_\alpha^+ \tau^{-\dot{\alpha}} + (\bar{\theta}^+)^2 (\bar{\theta}^-)^2 \tau^{--}, \end{aligned} \quad (6.45)$$

as a solution of the zero-curvature equation (4.13). All the component fields in the r.h.s. of (6.45) depend only on the variables  $\theta_\alpha^+$ ,  $\theta_\alpha^-$ . Substituting (6.44), (6.45) into (4.13), we find

$$v^{--} = (\theta^-)^2 \frac{\bar{\phi}}{1 + 4I\bar{\phi}}, \quad (6.46)$$

$$v^{-\dot{\alpha}} = \frac{2(\theta^- \sigma_m)^{\dot{\alpha}} A_m}{1 + 4I\bar{\phi}}, \quad (6.47)$$

$$\mathcal{A} = \phi + \frac{4I A_m A_m}{1 + 4I\bar{\phi}} + (\theta^+ \sigma_{mn} \theta^-) F_{mn}, \quad (6.48)$$

$$\tau^{-\dot{\alpha}} = -\frac{4I(\theta^- \sigma_{mn})^\alpha F_{mn} \sigma_{r\alpha}^{\dot{\alpha}} A_r}{1 + 4I\bar{\phi}}, \quad (6.49)$$

$$\tau^{--} = \varphi^{--} = \varphi_{mn}^{--} = 0. \quad (6.50)$$

Using the definitions (3.4), we find the component structure of the superfield strengths

$$W = \phi + \frac{4IA_mA_m}{1+4I\bar{\phi}} + (\theta^+\sigma_{mn}\theta^-)F_{mn} - 4I(\theta^-\sigma_{mn}^\alpha)(\sigma_r\bar{\theta}^+)_\alpha \frac{A_r F_{mn}}{1+4I\bar{\phi}}, \quad (6.51)$$

$$\bar{W} = \frac{\bar{\phi}}{1+4I\bar{\phi}} + (\bar{\theta}^+\bar{\sigma}_{mn}\bar{\theta}^-)\frac{F_{mn}}{1+4I\bar{\phi}} + (\bar{\theta}^+)^2(\bar{\theta}^-)^2\frac{2IF_{mn}F_{mn}+4IF_{mn}\tilde{F}_{mn}}{1+4I\bar{\phi}}. \quad (6.52)$$

Note that the superfields  $W$  and  $\bar{W}$  are deformed differently. Moreover, the superfield  $\bar{W}$  given by (6.52) does not depend on the  $x^m$  and  $\theta_\alpha^i$  variables, which agrees with the constraint (6.41).

Introducing the notations

$$\Phi = \phi + \frac{4IA_mA_m}{1+4I\bar{\phi}}, \quad (6.53)$$

we bring the superfield strength (6.51) to the standard form,

$$W = \Phi + (\theta^+\sigma_{mn}\theta^-)F_{mn} + \dots, \quad (6.54)$$

where dots correspond to the last term in (6.51) which does not contribute to the holomorphic effective action.

Now we substitute the superfield strength (6.54) into the holomorphic potential (6.40), compute the  $\star$ -products and integrate over Grassmann variables. As a result, we arrive at the following component expression for the holomorphic effective action

$$\Gamma_{hol} = -\frac{1}{32\pi^2} \int d^4x (F^2 + F\tilde{F}) \left[ \ln \frac{\Phi}{\mu} + \Delta(X(\Phi, F_{mn})) \right], \quad (6.55)$$

where

$$\Delta(X) = \frac{1}{2}(1-X)^2 \ln(X-1) + \frac{1}{2}(1+X)^2 \ln(1+X) - (1+X^2) \ln X, \quad (6.56)$$

$$X(\Phi, F_{mn}) = \frac{\Phi}{2I\sqrt{2(F^2 + F\tilde{F})}}. \quad (6.57)$$

The function  $\Delta(X)$  in (6.55) is responsible for the non-anticommutative corrections to the standard terms in the holomorphic potential. In the limit  $I \rightarrow 0$  we have

$$\lim_{I \rightarrow 0} \Delta(X) = \frac{3}{2}. \quad (6.58)$$

We see here that (6.58) reproduces the constant  $3/2$  in (6.6). This constant was not essential in the undeformed case, but now it is replaced by the function  $\Delta(X)$ .

Let us finally study the non-anticommutative corrections in the antiholomorphic sector. We substitute the superfield strength (6.52) into the antiholomorphic potential (6.43) and integrate there over the Grassmann variables. As a result, we find the component structure of the antiholomorphic effective action,

$$\begin{aligned} \Gamma_{antihol} = & -\frac{1}{32\pi^2} \int d^4x \frac{(F^2 + F\tilde{F})}{(1+4I\bar{\phi})^2} \left( \ln \frac{\bar{\phi}}{\mu(1+4I\bar{\phi})} + \frac{3}{2} \right) \\ & -\frac{1}{32\pi^2} \int d^4x \frac{F^2 + 2F\tilde{F}}{(1+4I\bar{\phi})^2} 2I\bar{\phi} \left( 1 + 2 \ln \frac{\bar{\phi}}{\mu(1+4I\bar{\phi})} \right). \end{aligned} \quad (6.59)$$

In the limit  $I \rightarrow 0$  the expression (6.59) reproduces the standard antiholomorphic potential in the undeformed charged hypermultiplet theory.

## 7 Conclusions

In this review we considered  $\mathcal{N}=(1,0)$  non-anticommutative theories with a chiral singlet  $Q$ -deformation of  $\mathcal{N}=(1,1)$  supersymmetry in harmonic superspace. In particular, we studied abelian models of the gauge superfield and hypermultiplets, both classical and quantum. Let us give a brief summary of the basic results of the review.

In the superfield approach the non-anticommutative deformation of  $\mathcal{N}=(1,1)$  supersymmetry is taken into account by introducing a  $\star$ -product in  $\mathcal{N}=(1,1)$  superspace. The chiral singlet deformation of  $\mathcal{N}=(1,1)$  harmonic superspace is a particular case. Owing to the fact that the operation of  $\star$ -multiplication is compatible with the harmonic and Grassmann harmonic analyticities, classical actions of the gauge superfield and hypermultiplet models can be obtained simply by substituting the  $\star$ -product for the ordinary local product in the undeformed superfield actions. At the component level, the  $\star$ -products induce a modification of the actions by new terms proportional to the deformation parameter. We have presented the component structure of the deformed classical actions for the abelian models of the neutral and charged hypermultiplets, as well as for the gauge supermultiplet.

The quantum aspects of these non-anticommutative models are remarkable. The deformation parameter has negative mass dimension, so counterterms are expected to destroy the renormalizability. Nevertheless, we proved the renormalizability of the deformed models of the abelian gauge superfield and neutral hypermultiplet. It turned out that the divergent contributions to the effective action can be eliminated altogether by an appropriate shift of the scalar field  $\phi$ , one of two independent scalar fields present in the Euclidean  $\mathcal{N}=(1,1)$  vector gauge supermultiplet. This field redefinition has no impact on the quantum dynamics of the theory, and the theories under considerations are actually finite. The renormalizability of the abelian non-anticommutative model of the charged hypermultiplet was proved using perturbative quantum calculations in  $\mathcal{N}=(1,1)$  harmonic superspace. It turns out that the divergent part of the effective action is proportional to the superfield action of the  $\mathcal{N}=(1,0)$  model of the gauge superfield. In this sense the charged hypermultiplet model is also renormalizable. Moreover, in this model the holomorphic potential was calculated and found to follow from its undeformed analog just by employing  $\star$ -product universally. At the level of component fields this leads to new terms in the effective action which, at least in the bosonic sector, can be accommodated in the single function  $\Delta(X)$  defined in (6.56).

Summarizing, we point out that all theories with chiral deformations of  $\mathcal{N}=(1/2, 1/2)$  and  $\mathcal{N}=(1,1)$  supersymmetry studied so far are renormalizable. One naturally conjecture that any deformation of this type preserves the renormalizability properties. Surely, this hypothesis requires a rigorous proof and to be supported by further examples. In this connection, it would be interesting to prove the renormalizability of non-abelian  $\mathcal{N}=(1,0)$  non-anticommutative gauge theories with and without hypermultiplets, as well as to attack the problem of constructing the low-energy effective action in these theories.

An important problem for further study is the question of renormalizability and finiteness of non-anticommutative  $\mathcal{N}=4$  (actually,  $\mathcal{N}=(2,2)$  in Euclidean space) supersymmetric gauge theory corresponding to the chiral singlet deformation (2.25) in harmonic superspace [43, 29, 44, 45]. If the chiral deformations of supersymmetry preserve the finiteness of this model, the quantum aspects of such a deformed  $\mathcal{N}=4$  supergauge theory shall be very special.

Another possible direction of future investigation concerns the quantum study of non-singlet

deformations of  $\mathcal{N}=(1,1)$  supersymmetry. In particular, it is of interest to consider those deformations which reduce to the known deformations of  $\mathcal{N}=(1/2,1/2)$  supersymmetry upon the appropriate reduction of the Grassmann sector of  $\mathcal{N}=(1,1)$  superspace. In this way, one might compare the results of calculations in the  $\mathcal{N}=1$  and  $\mathcal{N}=2$  superfield approaches. Non-anticommutative models with non-singlet deformations of  $\mathcal{N}=(1,1)$  supersymmetry were considered at the classical level in [23].

To conclude, theories with non-anticommutative deformations of supersymmetry represent a prospective area for further studies. Only a small part of this new “continent” of applications of supersymmetry has been developed until today.

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## Appendices

### 1 Euclidean $\mathcal{N}=(1,1)$ superspace

The Euclidean  $\mathcal{N}=(1,1)$  superspace is defined as a superspace parametrized by the coordinates

$$z = \{x^m, \theta_i^\alpha, \bar{\theta}^{\dot{\alpha}i}\}, \quad \alpha, \dot{\alpha} = 1, 2, \quad i = 1, 2. \quad (\text{A.1})$$

Here  $\theta_i^\alpha, \bar{\theta}^{\dot{\alpha}i}$  are analytic Grassmann variables,  $x^m = (x^1, x^2, x^3, x^4)$  are coordinates of the Euclidean space  $\mathbb{R}^4$  with the metrics  $g_{mn} = \delta_{mn}$ . Since the metrics is given by the unit matrix  $\delta_{mn}$ , the objects with upper and lower indices are equivalent. Therefore, throughout this work we use only the vectors and tensors with lower indices, except for  $x^m$ , and the contraction over repeated indices is assumed.

The spinor  $\text{SU}(2)$  indices  $\alpha, \dot{\alpha}$  are raised and lowered with the antisymmetric  $\varepsilon$ -tensor,

$$\begin{aligned} \psi_\alpha &= \varepsilon_{\alpha\beta} \psi^\beta, & \bar{\psi}_{\dot{\alpha}} &= \varepsilon_{\dot{\alpha}\dot{\beta}} \bar{\psi}^{\dot{\beta}}, \\ \varepsilon_{12} &= -\varepsilon^{12} = \varepsilon_{\dot{1}\dot{2}} = -\varepsilon^{\dot{1}\dot{2}} = 1, & \varepsilon^{\alpha\beta} \varepsilon_{\beta\gamma} &= \delta_\gamma^\alpha, \quad \varepsilon^{\dot{\alpha}\dot{\beta}} \varepsilon_{\dot{\beta}\dot{\gamma}} = \delta_{\dot{\gamma}}^{\dot{\alpha}}. \end{aligned} \quad (\text{A.2})$$

We use the following conventions for the Euclidean sigma-matrices

$$\begin{aligned} (\sigma_m)_{\alpha\dot{\alpha}} &= (i\vec{\sigma}, \mathbf{1})_{\alpha\dot{\alpha}}, & (\bar{\sigma}_m)^{\dot{\alpha}\alpha} &= \varepsilon^{\alpha\beta} \varepsilon^{\dot{\alpha}\dot{\beta}} (\sigma_m)_{\beta\dot{\beta}}, \\ \sigma_m \bar{\sigma}_n + \sigma_n \bar{\sigma}_m &= 2\delta_{mn}, & \sigma_{mn} &= \frac{i}{2} (\sigma_m \bar{\sigma}_n - \sigma_n \bar{\sigma}_m), \\ \text{tr } \sigma_n \bar{\sigma}_m &= 2\delta_{mn}, & \text{tr } (\sigma_n \bar{\sigma}_m \sigma_p \bar{\sigma}_r) &= 2\delta_{mn} \delta_{pr} - 2\delta_{np} \delta_{mr} + \delta_{nr} \delta_{pm} - 2\varepsilon_{nmpr}, \end{aligned} \quad (\text{A.3})$$

where  $\vec{\sigma}$  are the Pauli matrices.

Supercharges and covariant spinor derivatives in  $\mathcal{N}=(1,1)$  superspace are given by

$$\begin{aligned} Q_\alpha^i &= \partial_\alpha^i - i\bar{\theta}^{\dot{\alpha}i}(\sigma_m)_{\alpha\dot{\alpha}} \frac{\partial}{\partial x^m}, & \bar{Q}_{\dot{\alpha}i} &= -\bar{\partial}_{\dot{\alpha}i} + i\theta_i^\alpha(\sigma_m)_{\alpha\dot{\alpha}} \frac{\partial}{\partial x^m}, \\ D_\alpha^i &= \partial_\alpha^i + i\bar{\theta}^{\dot{\alpha}i}(\sigma_m)_{\alpha\dot{\alpha}} \frac{\partial}{\partial x^m}, & \bar{D}_{\dot{\alpha}i} &= -\bar{\partial}_{\dot{\alpha}i} - i\theta_i^\alpha(\sigma_m)_{\alpha\dot{\alpha}} \frac{\partial}{\partial x^m}, \end{aligned} \quad (\text{A.4})$$

where the anticommuting derivatives  $\partial_\alpha^i = \frac{\partial}{\partial \theta_i^\alpha}$ ,  $\bar{\partial}_{\dot{\alpha}i} = \frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}i}}$  act on the Grassmann variables by the rules

$$\partial_\alpha^i \theta_j^\beta = \delta_j^i \delta_\alpha^\beta, \quad \bar{\partial}_{\dot{\alpha}i} \bar{\theta}^{\dot{\beta}j} = \delta_i^j \delta_{\dot{\alpha}}^{\dot{\beta}}. \quad (\text{A.5})$$

The non-vanishing anticommutation relations between the operators (A.4) are as follows

$$\{D_\alpha^i, \bar{D}_{\dot{\alpha}j}\} = -\{Q_\alpha^i, \bar{Q}_{\dot{\alpha}j}\} = -2i\delta_j^i(\sigma_m)_{\alpha\dot{\alpha}} \frac{\partial}{\partial x^m}. \quad (\text{A.6})$$

## 2 Euclidean harmonic superspace

The harmonic variables  $u_i^\pm$ ,  $i = 1, 2$ , are defined as the coordinates parametrizing the coset  $\text{SU}(2)/\text{U}(1)$  and obeying the following basic relations

$$u^{\pm k} = \varepsilon^{kj} u_j^\pm, \quad u^{+k} u_k^- = 1. \quad (\text{A.7})$$

The harmonic variables (A.7) allow ones to convert the internal symmetry group indices into the  $\text{U}(1)$  indices  $\pm$ , e.g.,

$$\begin{aligned} \theta^{\pm\alpha} &= \theta^{\alpha k} u_k^\pm, & \bar{\theta}^{\pm\dot{\alpha}} &= \bar{\theta}^{\dot{\alpha} k} u_k^\pm, \\ Q_\alpha^\pm &= Q_\alpha^k u_k^\pm, & \bar{Q}_{\dot{\alpha}}^\pm &= \bar{Q}_{\dot{\alpha}}^k u_k^\pm, & D_\alpha^\pm &= D_\alpha^k u_k^\pm, & \bar{D}_{\dot{\alpha}}^\pm &= \bar{D}_{\dot{\alpha}}^k u_k^\pm. \end{aligned} \quad (\text{A.8})$$

A key feature of the superspace with the coordinates  $(x^m, \theta^{\pm\alpha}, \bar{\theta}^{\pm\dot{\alpha}}, u^{\pm i})$  is the presence of the so called analytic subspace  $(\zeta, u) = (x_A^m, \theta^+, \bar{\theta}^+, u^{\pm i})$ , where

$$x_A^m = x^m - i(\theta^{+\alpha}(\sigma^m)_{\alpha\dot{\alpha}} \bar{\theta}^{-\dot{\alpha}} + \theta^{-\alpha}(\sigma^m)_{\alpha\dot{\alpha}} \bar{\theta}^{+\dot{\alpha}}). \quad (\text{A.9})$$

The analytic subspace is closed under supersymmetry,

$$\delta_\epsilon x_A^m = -2i(\sigma^m)_{\alpha\dot{\alpha}}(\epsilon^{-\alpha} \bar{\theta}^{+\dot{\alpha}} + \theta^{+\alpha} \bar{\epsilon}^{-\dot{\alpha}}), \quad \delta_\epsilon \theta^{\pm\alpha} = \epsilon^{\pm\alpha}, \quad \delta_\epsilon \bar{\theta}^{\pm\dot{\alpha}} = \bar{\epsilon}^{\pm\dot{\alpha}}. \quad (\text{A.10})$$

Here  $\epsilon^{\pm\alpha} = \epsilon^{\alpha k} u_k^\pm$ ,  $\bar{\epsilon}^{\pm\dot{\alpha}} = \bar{\epsilon}^{\dot{\alpha} k} u_k^\pm$  and  $\epsilon^{\alpha k}$ ,  $\bar{\epsilon}^{\dot{\alpha} k}$  are anticommuting parameters of supertranslations. Therefore there exist the so called analytic superfields which “live” on the subset of analytic coordinates,  $\Phi_A = \Phi_A(\zeta, u)$ . Such superfields are singled out from the general superfields on  $\mathcal{N}=(1,1)$  superspace by the following covariant analyticity (Grassmann Cauchy-Riemann) conditions

$$D_\alpha^+ \Phi_A = 0, \quad \bar{D}_{\dot{\alpha}}^+ \Phi_A = 0, \quad (\text{A.11})$$

where the covariant spinor derivatives  $D_\alpha^\pm$ ,  $\bar{D}_{\dot{\alpha}}^\pm$  in the analytic basis are given by

$$\begin{aligned} D_\alpha^+ &= \frac{\partial}{\partial \theta^{+\alpha}}, & D_\alpha^- &= -\frac{\partial}{\partial \theta^{+\alpha}} + 2i(\sigma_m)_{\alpha\dot{\alpha}} \bar{\theta}^{-\dot{\alpha}} \frac{\partial}{\partial x_A^m}, \\ \bar{D}_{\dot{\alpha}}^+ &= \frac{\partial}{\partial \bar{\theta}^{+\dot{\alpha}}}, & \bar{D}_{\dot{\alpha}}^- &= -\frac{\partial}{\partial \bar{\theta}^{+\dot{\alpha}}} - 2i(\sigma_m)_{\alpha\dot{\alpha}} \theta^{-\alpha} \frac{\partial}{\partial x_A^m}. \end{aligned} \quad (\text{A.12})$$

In the harmonic superspace approach the harmonic variables  $u_i^\pm$  are considered on equal footing with the Grassmann and space-time ones. In particular, there are covariant harmonic derivatives, which in the analytic coordinates  $(x_A^m, \theta_\alpha^\pm, \bar{\theta}_{\dot{\alpha}}^\pm, u)$  are given by

$$\begin{aligned} D^{++} &= u_i^+ \frac{\partial}{\partial u_i^-} - 2i\theta^{+\alpha}(\sigma_m)_{\alpha\dot{\alpha}}\bar{\theta}^{+\dot{\alpha}} \frac{\partial}{\partial x_A^m} + \theta_\alpha^+ \frac{\partial}{\partial \theta_\alpha^-} + \bar{\theta}_{\dot{\alpha}}^+ \frac{\partial}{\partial \bar{\theta}_{\dot{\alpha}}^-}, \\ D^{--} &= u_i^- \frac{\partial}{\partial u_i^+} - 2i\theta^{-\alpha}(\sigma_m)_{\alpha\dot{\alpha}}\bar{\theta}^{-\dot{\alpha}} \frac{\partial}{\partial x_A^m} + \theta_\alpha^- \frac{\partial}{\partial \theta_\alpha^+} + \bar{\theta}_{\dot{\alpha}}^- \frac{\partial}{\partial \bar{\theta}_{\dot{\alpha}}^+}, \\ D^0 &= [D^{++}, D^{--}] = u_i^+ \frac{\partial}{\partial u_i^+} - u_i^- \frac{\partial}{\partial u_i^-} + \theta_\alpha^+ \frac{\partial}{\partial \theta_\alpha^+} + \bar{\theta}_{\dot{\alpha}}^+ \frac{\partial}{\partial \bar{\theta}_{\dot{\alpha}}^+} - \theta_\alpha^- \frac{\partial}{\partial \theta_\alpha^-} - \bar{\theta}_{\dot{\alpha}}^- \frac{\partial}{\partial \bar{\theta}_{\dot{\alpha}}^-}. \end{aligned} \quad (\text{A.13})$$

The derivatives (A.13) obey the commutation relations of the algebra  $\text{su}(2)$ .

The integration over the Grassmann and harmonic variables is defined by the rules

$$\begin{aligned} \int d^4\theta(\theta^+)^2(\theta^-)^2 &= 1, \quad \int d^4\theta^-(\theta^+)^2(\bar{\theta}^+)^2 = 1, \quad \int d^8\theta(\theta^+)^2(\bar{\theta}^+)^2(\theta^-)^2(\bar{\theta}^-)^2 = 1, \\ \int du \, 1 &= 1, \quad \int du \, u^{+(i_1} \dots u^{+i_n} u^{-j_1} \dots u^{-j_m)} = 0. \end{aligned} \quad (\text{A.14})$$

Here we use the following notation

$$(\theta^+)^2 = \theta^{+\alpha}\theta_\alpha^+, \quad (\bar{\theta}^+)^2 = \bar{\theta}_{\dot{\alpha}}^+\bar{\theta}^{+\dot{\alpha}}, \quad \theta^-\sigma_{mn}\theta^+ = \theta^{-\alpha}(\sigma_{mn})_\alpha^\beta\theta_\beta^+ = -\theta^+\sigma_{mn}\theta^-. \quad (\text{A.15})$$

We use also the chiral-analytic coordinates  $Z_C = (z_C, \bar{\theta}^{\pm\dot{\alpha}})$ , where

$$z_C = (x_L^m, \theta^{\pm\alpha}). \quad (\text{A.16})$$

The covariant spinor and harmonic derivatives, as well as the  $\mathcal{N}=(1,0)$  supercharges, in these coordinates read

$$\begin{aligned} D_\alpha^+ &= \partial_{-\alpha} + 2i\bar{\theta}^{+\dot{\alpha}}\partial_{\alpha\dot{\alpha}}, & D_\alpha^- &= -\partial_{+\alpha} + 2i\bar{\theta}^{-\dot{\alpha}}\partial_{\alpha\dot{\alpha}}, \\ \bar{D}_{\dot{\alpha}}^+ &= \bar{\partial}_{-\dot{\alpha}}, & \bar{D}_{\dot{\alpha}}^- &= -\bar{\partial}_{+\dot{\alpha}} \end{aligned} \quad (\text{A.17})$$

$$\begin{aligned} D_C^{++} &= \partial^{++} + \theta^{+\alpha}\partial_{-\alpha} + \bar{\theta}^{+\dot{\alpha}}\bar{\partial}_{-\dot{\alpha}}, \\ D_C^{--} &= \partial^{--} + \theta^{-\alpha}\partial_{+\alpha} + \bar{\theta}^{-\dot{\alpha}}\bar{\partial}_{+\dot{\alpha}}, \end{aligned} \quad (\text{A.18})$$

$$Q_\alpha^+ = \partial_{-\alpha}, \quad Q_\alpha^- = -\partial_{+\alpha}, \quad (\text{A.19})$$

where  $\partial^{++} = u_i^+ \frac{\partial}{\partial u_i^-}$ ,  $\partial^{--} = u_i^- \frac{\partial}{\partial u_i^+}$ . An analytic superfield  $\Lambda(\zeta, u)$  can be represented in the chiral-analytic coordinates as

$$\Lambda(\zeta, u) = \Lambda(\zeta_C, u) - 2i(\theta^-\sigma_m\bar{\theta}^+)\partial_m\Lambda(\zeta_C, u) - (\theta^-)^2(\bar{\theta}^+)^2\Box\Lambda(\zeta_C, u), \quad (\text{A.20})$$

where  $\zeta_C = (x_L^m, \theta^+, \bar{\theta}^+)$  and the component fields in the  $\theta$  and harmonic expansion of  $\Lambda(\zeta_C, u)$  depend on the coordinates  $x_L^m$ .

For the antichiral superfields we use also the antichiral coordinates,

$$x_R^m = x_A^m + 2i\theta^+\sigma^m\bar{\theta}^- = x_L^m + 2i\theta^+\sigma^m\bar{\theta}^- - 2i\theta^-\sigma^m\bar{\theta}^+. \quad (\text{A.21})$$

The  $\mathcal{N}=(1,0)$  supercharges and the covariant spinor derivatives in these coordinates are given by the expressions

$$\begin{aligned} Q_\alpha^+ &= \partial_{-\alpha} - 2i\bar{\theta}^{+\dot{\alpha}}\partial_{\alpha\dot{\alpha}}, & Q_\alpha^- &= -\partial_{+\alpha} - 2i\bar{\theta}^{-\dot{\alpha}}\partial_{\alpha\dot{\alpha}}, \\ D_\alpha^+ &= \partial_{-\alpha}, & D_\alpha^- &= -\partial_{+\alpha}. \end{aligned} \quad (\text{A.22})$$



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